

## ON THE REAL DIFFERENTIAL OF A SLICE REGULAR FUNCTION

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ABSTRACT. In this paper we show that the real differential of any injective slice regular function is everywhere invertible. The result is a generalization of a theorem proved by G. Gentili, S. Salamon and C. Stoppato, and it is obtained thanks, in particular, to some new information regarding the first coefficients of a certain polynomial expansion for slice regular functions (called *spherical expansion*), and to a new general result which says that the slice derivative of any injective slice regular function is different from zero. A useful tool proven in this paper is a new formula that relates slice and spherical derivatives of a slice regular function. Given a slice regular function, part of its singular set is described as the union of surfaces on which it results to be constant.

function of a hypercomplex variable and slice regular functions and power series and differential forms

## 1. INTRODUCTION

In [20] and [7], the authors start an interesting investigation about the real differential of a slice regular function (see also [10], Chapter 8.5). Let us firstly describe in few words the actors of this story; in the next section, everything will be properly formalized.

Denoting by  $\mathbb{H}$  the real algebra of quaternions and by  $\mathbb{S} \subset \mathbb{H}$  the subset of imaginary units

$$\mathbb{S} := \{q \in \mathbb{H} \mid q^2 = -1\},$$

we can write any quaternion as  $x = \alpha + I\beta$ , where  $\alpha$  and  $\beta$  are real numbers and  $I \in \mathbb{S}$ .

A *slice function*  $f : \Omega \subseteq \mathbb{H} \rightarrow \mathbb{H}$  is a quaternionic function of one quaternionic variable that is  $\mathbb{H}$ -left affine with respect to the imaginary unit, i.e. such that, for each  $x = \alpha + I\beta \in \Omega$ , it holds,

$$f(x) = F_1(\alpha, \beta) + IF_2(\alpha, \beta),$$

where,  $F_1$  and  $F_2$  satisfy an additional technical requirement.

A *slice regular function* is a slice function  $f$  such that, for any  $I \in \mathbb{S}$  the restriction of  $f$  to the complex line  $\mathbb{C}_I := \text{span}_{\mathbb{R}}\{1, I\} \subset \mathbb{H}$  is a holomorphic map.

The theory founded on this notion of regularity, introduced by Cullen in [6], is rapidly growing in the last years, thanks firstly to the authors of [4, 11, 12], who have set down the groundwork.

The main purpose of this paper is to *extend*, by removing the hypothesis concerning the domain, the next theorem stated in [7].

**Theorem 1** ([7], Corollary 3.10). *Let  $f : \Omega \rightarrow \mathbb{H}$  be an injective slice regular function with  $\Omega \cap \mathbb{R} \neq \emptyset$ . Then its real differential is everywhere invertible.*

Many results about slice regular functions defined over domains that intersects the real line, does not extend in an automatic way to functions defined over domains that do not have real points (see e.g. [2, 14, 15, 16]). Even in this case, the proof of Theorem 1 contained in [7] can not be adapted to our setting. To obtain the goal, we apply results from [7, 16, 20], taking into account the mentioned difference.

In particular, we firstly realize that a part of the theory could be formalized in a new way considering the *slice factor* of the real differential of a slice function: we introduce here the

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2010 *Mathematics Subject Classification.* 30G35, 30B10, 58A10 .

*Key words and phrases.* function of a hypercomplex variable; slice regular functions; differential forms.

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concept of *slice differential* of a slice function. After that, we remember the notion of *spherical analyticity* introduced in [20] for functions with domain intersecting the real axis, and in [16] for functions defined over any domain (in a more general context). We give some new information about the first coefficients of the spherical expansion of a slice regular function, showing a new way to compute slice derivatives (see formula 3). Finally, starting from some results about the rank of the real differential of a slice regular function, we extend Theorem 1, using, moreover, a new proposition that generalizes, in our context, a classical theorem of complex analysis (see Theorem 32). While describing these materials we show that a slice regular function can be constant either globally or on sets of real dimension two. We start here an investigation about the geometry of such surfaces.

The structure of the present work is the following.

In Section 2 we state the main preliminary results needed for the reader to understand the theory. At the end of this section are showed new results regarding the sets on which a slice regular function can be constant. For the non-original material, we mention, as general references for this part, the two books [5, 10] and the paper [14].

In Section 3 we introduce the concept of slice differential of a slice function.

Section 4 is divided into 3 subsections: an introductory one in which we describe the real differential of a slice regular function; in the second part we remember the main results about spherical analyticity and we discuss about the first coefficients of this expansion; in the last one, we deal with the rank of the real differential of a slice regular function.

Most of the results presented in this paper are contained in the Ph.D. thesis of the author [1].

## 2. PRELIMINARY RESULTS

We introduce, in this section, some notion developed by R. Ghiloni and A. Perotti (see [14, 15, 16]). In  $\mathbb{H}$  we will denote with  $x^c$  the usual conjugation, i.e.: if  $x = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$ , then  $x^c = x_0 - ix_1 - jx_2 - kx_3$ . Let  $\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$  be the complexification of  $\mathbb{H}$ . An element  $x$  in  $\mathbb{H}_{\mathbb{C}}$  will be of the form  $x = p + \sqrt{-1}q$ , where  $p$  and  $q$  are quaternions. The space  $\mathbb{H}_{\mathbb{C}}$  is a complex alternative algebra with a unity with respect to the product defined by the formula

$$(x + \sqrt{-1}y)(z + \sqrt{-1}w) := xz - yw + \sqrt{-1}(xw + yz).$$

In  $\mathbb{H}_{\mathbb{C}}$  we have two commuting antiinvolution acting on the element  $x$  in the following ways:

- $x^c = (p + \sqrt{-1}q)^c = p^c + \sqrt{-1}q^c$ ,
- $\bar{x} = (p + \sqrt{-1}q) = p - \sqrt{-1}q$

Given now  $D$  a connected open set in  $\mathbb{C}$ , we remember the following definitions from [14].

**Definition 1** ([14]). A function  $F : D \rightarrow \mathbb{H}_{\mathbb{C}}$  is called a *stem function* on  $D$  if it is complex intrinsec, i.e.: if the condition  $F(\bar{z}) = \overline{F(z)}$  holds for each  $z \in D$  such that  $\bar{z} \in D$ . Moreover we say that  $F = F_1 + \sqrt{-1}F_2$  has a certain regularity (e.g.:  $\mathcal{C}^0$ ,  $\mathcal{C}^k$ ,  $\mathcal{C}^\omega$ , etc), if the two components  $F_1$  and  $F_2$  have that regularity.

The last definition means that if  $D$  is symmetric with respect to the real axis and  $F_1, F_2 : D \rightarrow \mathbb{H}$  are the quaternionic components of  $F = F_1 + \sqrt{-1}F_2$  then  $F_1$  is even with respect to the imaginary part of  $z$  (i.e.:  $F_1(\bar{z}) = F_1(z)$ ), while  $F_2$  is odd (i.e.:  $F_2(\bar{z}) = -F_2(z)$ ). Thanks to this fact, there is no loss of generality in requiring  $D \subset \mathbb{C}$  to be symmetric with respect to the real axis.

**Definition 2** ([14]). Given any set  $D \subset \mathbb{C}$  we define the *circularization* of  $D$  in  $\mathbb{H}$  as the subset of  $\mathbb{H}$  defined by:

$$\Omega_D := \{\alpha + J\beta \in \mathbb{H} \mid \alpha + i\beta \in D, J \in \mathbb{S}\}.$$

Moreover any set of this kind will be called a *circular set*.

**Remark 1.** If  $D$  is a domain in  $\mathbb{C}$  such that  $D \cap \mathbb{R} \neq \emptyset$ , then  $\Omega_D$  is a *slice domain*.

We will use the following notations: let  $D$  be a subset of the complex plane  $\mathbb{C}$ , then we will denote with  $D_J$  and  $D_J^+$  the following sets:

$$D_J := \Omega_D \cap \mathbb{C}_J, \quad D_J^+ := \Omega_D \cap \mathbb{C}_J^+, \quad \forall J \in \mathbb{S},$$

where  $\mathbb{C}_J := \{\alpha + J\beta \in \mathbb{H} \mid \alpha + i\beta \in \mathbb{C}, J \in \mathbb{S}\}$  and  $\mathbb{C}_J^+ := \{\alpha + J\beta \in \mathbb{C}_J \mid \beta \geq 0\}$ . Moreover, we will call  $D_J$  and  $D_J^+$  a *slice* and *semislice* of  $\Omega_D$ , respectively.

**Remark 2.** If  $D \cap \mathbb{R} = \emptyset$ , then  $\Omega_D \simeq D^+ \times \mathbb{S}$  and so  $D_J \simeq D \times \{-J, J\}$ , while  $D_J^+ \simeq D \times \{J\}$ .

**Definition 3** ([14]). A function  $f : \Omega_D \rightarrow \mathbb{H}$  is called a *(left) slice function* if it is induced by a stem function  $F = F_1 + \sqrt{-1}F_2$  on  $D$ ,  $f = \mathcal{I}(F)$ , in the following way:

$$f(\alpha + J\beta) := F_1(\alpha + i\beta) + JF_2(\alpha + i\beta), \quad \forall x = \alpha + J\beta \in \Omega_D.$$

We will denote by  $\mathcal{S}(\Omega_D)$  and by  $\mathcal{S}^k(\Omega_D)$ , for any  $k \in \mathbb{N} \cup \{\infty\}$ , the real vector spaces and right  $\mathbb{H}$ -module of slice functions on  $\Omega_D$  induced by continuous and of class  $\mathcal{C}^k$  stem functions, respectively. Thanks to Definition 1, any slice function is well defined. Indeed, if  $D$  is symmetric with respect to the real axis and  $f = \mathcal{I}(F) : \Omega_D \rightarrow \mathbb{H}$  is a slice function induced by  $F$ , then  $f(\alpha + (-J)(-\beta)) = F_1(\alpha + i(-\beta)) - JF_2(\alpha + i(-\beta)) = F_1(\alpha + i\beta) - J(-F_2(\alpha + i\beta)) = F_1(\alpha + i\beta) + JF_2(\alpha + i\beta) = f(\alpha + J\beta)$ .

For slice functions we have the following representation theorem. It says that if we know the values of a slice function over two different semislices then we can reconstruct the whole function. This result is not surprising having in mind the “affine nature” of a slice function with respect to the imaginary unit. The precise statement is the following one.

**Theorem 2** ([14], Proposition 6). *Let  $f$  be a slice function on  $\Omega_D$ . If  $J \neq K \in \mathbb{S}$  then, for every  $x = \alpha + I\beta \in \Omega_D$ , the following formula holds:*

$$f(x) = (I - K)(J - K)^{-1}f(\alpha + J\beta) - (I - J)(J - K)^{-1}f(\alpha + K\beta).$$

In particular, for  $K = -J$ , we get the formula

$$f(x) = \frac{1}{2}(f(\alpha + J\beta) + f(\alpha - J\beta) - IJ(f(\alpha + J\beta) - f(\alpha - J\beta))).$$

Representation formulas for quaternionic slice regular functions appeared in [3, 4], while the case of continuous slice functions can be found in [14].

**Definition 4** ([14]). Given a slice function  $f$ , we define its spherical derivative in  $x \in \Omega_D \setminus \mathbb{R}$  as,

$$\partial_s f(x) := \frac{1}{2}Im(x)^{-1}(f(x) - f(x^c)).$$

**Remark 3.** We have that  $\partial_s f = \mathcal{I}(\frac{F_2(z)}{Im(z)})$  on  $\Omega_D \setminus \mathbb{R}$ . Given  $x = \alpha + J\beta \in \Omega_D$ , the spherical derivative is constant on the sphere

$$\mathbb{S}_x = \{y \in \mathbb{H} \mid y = \alpha + I\beta, I \in \mathbb{S}\}.$$

Moreover,  $\partial_s f = 0$  if and only if  $f$  is constant on  $\mathbb{S}_x$ , in other terms:

$$\partial_s(\partial_s(f)) = 0.$$

If  $\Omega_D \cap \mathbb{R} \neq \emptyset$ , under some mild regularity hypothesis on  $F$  (see [14], Proposition 7 for more details),  $\partial_s f$  can be extended continuously as a slice function on  $\Omega_D$ . In particular this is true if the stem function  $F$  is of class  $\mathcal{C}^1$ .

Let  $D \subset \mathbb{C}$  be an open set. Given a  $\mathcal{C}^1$  stem function  $F = F_1 + \sqrt{-1}F_2 : D \rightarrow \mathbb{H}_{\mathbb{C}}$ , the two functions

$$\frac{\partial F}{\partial z}, \frac{\partial F}{\partial \bar{z}} : D \rightarrow \mathbb{H}_{\mathbb{C}},$$

are stem functions too. Explicitly:

$$\frac{\partial F}{\partial z} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha} - \sqrt{-1} \frac{\partial F}{\partial \beta} \right) = \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - \sqrt{-1} \left( \frac{\partial F_1}{\partial \beta} - \frac{\partial F_2}{\partial \alpha} \right) \right),$$

and

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial F}{\partial \alpha} + \sqrt{-1} \frac{\partial F}{\partial \beta} \right) = \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + \sqrt{-1} \left( \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha} \right) \right).$$

The previous stem functions induce the continuous slice derivatives:

$$\frac{\partial f}{\partial x} := \mathcal{I} \left( \frac{\partial F}{\partial z} \right), \quad \frac{\partial f}{\partial x^c} := \mathcal{I} \left( \frac{\partial F}{\partial \bar{z}} \right).$$

While the spherical derivative controls the behavior of a slice function  $f$  along the “spherical” directions determined by  $\mathbb{S}$ , the slice derivatives  $\partial/\partial x$  and  $\partial/\partial x^c$ , give information about the behavior along the remaining directions (i.e.: along the (semi)slices).

If  $f = \mathcal{I}(F) : \Omega_D \rightarrow \mathbb{H}$ , then we denote the restrictions over a slice or a semislice, as

$$f_J := f|_{D_J} : D_J \rightarrow \mathbb{H}, \quad f_J^+ := f|_{D_J^+} : D_J^+ \rightarrow \mathbb{H},$$

respectively.

The following is a rewriting of a lemma contained in [16].

**Lemma 3** ([16], Lemma 2.1). *Let  $f \in \mathcal{S}^1(\Omega_D)$  and let  $J \in \mathbb{S}$ . Then, for each  $x = \alpha + J\beta \in \Omega_D$ , it holds:*

$$\frac{\partial f}{\partial x}(\alpha + J\beta) = \frac{\partial f_J}{\partial z_J}(\alpha + J\beta) \quad \text{and} \quad \frac{\partial f}{\partial x^c}(\alpha + J\beta) = \frac{\partial f_J}{\partial \bar{z}_J}(\alpha + J\beta),$$

where  $\partial/\partial z_J := (1/2)(\partial/\partial\alpha - J \cdot \partial/\partial\beta)$  and  $\partial/\partial \bar{z}_J := (1/2)(\partial/\partial\alpha + J \cdot \partial/\partial\beta)$ . Furthermore, if  $f \in \mathcal{S}^\infty(\Omega_D)$  and  $n \in \mathbb{N}$ , then

$$\frac{\partial^n f}{\partial x^n}(\alpha + J\beta) = \frac{\partial^n f_J}{\partial z_J^n}(\alpha + J\beta)$$

So, the slice derivatives at a certain point  $x = \alpha + J\beta$  of a slice function  $f$  can be computed by restricting the function to the proper semislice (in this case to  $\mathbb{C}_J$ ), and then deriving with respect to  $\partial/\partial z$  or  $\partial/\partial \bar{z}$ .

Now, left multiplication by  $\sqrt{-1}$  defines a complex structure on  $\mathbb{H}_{\mathbb{C}}$  and, with respect to this structure, a  $C^1$  stem function

$$F = F_1 + \sqrt{-1}F_2 : D \rightarrow \mathbb{H}_{\mathbb{C}}$$

is holomorphic if and only if satisfy the Cauchy-Riemann equations

$$\frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \quad \frac{\partial F_2}{\partial \beta} = -\frac{\partial F_1}{\partial \alpha}, \quad z = \alpha + i\beta \in D$$

or equivalently if

$$\frac{\partial F}{\partial \bar{z}} \equiv 0.$$

We are now in position to define slice regular functions.

**Definition 5** ([14]). A slice function  $f \in \mathcal{S}^1(\Omega_D)$  is called *slice regular* if the following equation holds:

$$\frac{\partial f}{\partial x^c}(\alpha + J\beta) = 0, \quad \forall \alpha + J\beta \in \Omega_D.$$

We denote by  $\mathcal{SR}(\Omega_D)$  the real vector space of all slice regular functions on  $\Omega_D$ .

**Remark 4.** Originally, a slice regular function was defined as a function  $f : \Omega_D \subseteq \mathbb{H} \rightarrow \mathbb{H}$  such that, for any  $I \in \mathbb{S}$ , the restriction  $f_I$  has continuous partial derivatives and  $\partial f_I / \partial \bar{z}$  vanishes identically (cf. [10], Definition 1.1). Anyway, if this definition implies *sliceness* when  $D \cap \mathbb{R} \neq \emptyset$ , this is no more true in the general case. Furthermore, in [15] it is shown that the class of quaternionic functions which are holomorphic if restricted to any complex line  $\mathbb{C}_I$  and are not *slice* is too big and hence not very manageable.

A slice regular function is, then, a slice function induced by a holomorphic stem function. The next theorem gives a characterization of slice regular functions.

**Proposition 4** ([14], Proposition 8 and Remark 6). *Let  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$ , then the following facts are equivalents:*

- $f \in \mathcal{SR}(\Omega_D)$ ;
- the restriction  $f_J$  is holomorphic for every  $J \in \mathbb{S}$  with respect to the complex structures on  $D_J$  and  $\mathbb{H}$  defined by left multiplication by  $J$ ;

- two restrictions  $f_J^+, f_K^+$  ( $J \neq K$ ) are holomorphic on  $D_J^+$  and  $D_K^+$  respectively (the possibility  $K = -J$  is not excluded).

Lemma 3, implies that, if the set  $D$  has nonempty intersection with the real line, then  $f$  is slice regular on  $\Omega_D$  if and only if it is Cullen regular in the sense introduced by Gentili and Struppa in [11, 12].

We recall that any slice regular function restricted to a slice admits a splitting into two complex holomorphic function as the following lemma claims. A proof of this result can be found in [4] or in [16]. In [4] the result is proven with the additional hypothesis that the domain of definition intersects the real axis.

**Lemma 5** ([4], Lemma 2.3). *Let  $f \in \mathcal{SR}(\Omega_D)$  and  $J \perp K$  two elements of  $\mathbb{S}$ . Then there exist two holomorphic functions  $f_1, f_2 : D_J \rightarrow \mathbb{C}_J$  such that*

$$f_J = f_1 + f_2 K.$$

The pointwise product of two slice functions is not, in general, a slice function<sup>1</sup> but, if one considers the function induced by the pointwise product of the two stem functions, then the result is a slice function and also regularity is preserved. So, to be more precise, we state the following definition.

**Definition 6** ([14]). Let  $f = \mathcal{I}(F)$ ,  $g = \mathcal{I}(G)$  be two slice functions on  $\Omega_D$ . The *slice product* of  $f$  and  $g$  is the slice function defined by

$$f \cdot g := \mathcal{I}(FG) = \mathcal{I}(F_1 G_1 - F_2 G_2 + \sqrt{-1}(F_1 G_2 + F_2 G_1)).$$

**Remark 5.** In the previous definition, if the components of the first stem function  $F = F_1 + \sqrt{-1}F_2$  are real valued, then  $(f \cdot g)(x) = f(x)g(x)$  for each  $x \in \Omega_D$ .

**Definition 7** ([14]). A slice function  $f = \mathcal{I}(F)$  is called *real*, if the two components  $F_1$  and  $F_2$  are real-valued.

The next proposition says that this notion of product is the good one, meaning that it preserves regularity.

**Proposition 6** ([14], Proposition 11). *If  $f, g \in \mathcal{SR}(\Omega_D)$ , then  $f \cdot g \in \mathcal{SR}(\Omega_D)$ .*

In [14] it is also pointed out that the regular product introduced in [4, 8] is generalized by this one if the domain  $\Omega_D$  does not have real points. In the next proposition we explicit the slice product as the pointwise product with the proper evaluations. This proposition was proved for regular functions defined on domains that intersect the real axis in [4, 9, 8] and in [2] in this general setting.

**Proposition 7** ([2], Proposition 4.8). *Let  $f, g \in \mathcal{SR}(\Omega_D)$  then, for any  $x \in \Omega_D \setminus V(f)$*

$$(f \cdot g)(x) = f(x)g(f(x)^{-1}xf(x)).$$

Some notion about the zero set of a slice regular function will be useful in the next sections. We will quote, then, the main results known in the literature.

If  $F$  is a stem function, then  $F^c$  is a stem function as well. We will denote by  $f^c$  the slice function induced by  $F^c$ . The next definition given in [14] generalizes the one given in [8] for power series.

**Definition 8** ([14]). Let  $f$  be a slice function over  $\Omega_D$ . Then we define the *normal function* of  $f$  (or *symmetrization* of  $f$ ) as the slice function  $N(f) := f \cdot f^c \in \mathcal{S}(\Omega_D)$ .

**Remark 6.** Let  $f$  be a slice function. The following facts are contained in [14], Section 6.

- If  $f$  is a slice regular function, then also  $f^c$  and  $N(f)$  are slice regular functions.

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<sup>1</sup>Take, for instance,  $a \in \mathbb{H} \setminus \mathbb{R}$ ,  $f(x) = xa$  and  $g(x) = x$ . Then, the lack of commutativity implies that,  $h(x) = f(x)g(x) = xax$  is not a slice function.

- The following equation holds true:

$$(f \cdot g)^c = g^c \cdot f^c, \quad \text{and so} \quad N(f) = N(f)^c.$$

Moreover,  $N(f^c) = N(f)$ .

- The next equality holds true:

$$N(f \cdot g) = N(f)N(g).$$

Let now  $V(f)$  be the zero set of  $f : \Omega_D \rightarrow \mathbb{H}$ :

$$V(f) := \{x \in \Omega_D \mid f(x) = 0\}.$$

The next proposition is a consequence of the “affine behavior” of slice regular functions with respect to imaginary units.

**Proposition 8** ([14], Proposition 16). *Let  $f \in \mathcal{S}(\Omega_D)$ , then, for any  $x \in \Omega_D \setminus \mathbb{R}$ , the restriction of  $f$  to  $\mathbb{S}_x$  is injective or constant.*

A structure result for  $V(f)$  is showed in the next theorem.

**Theorem 9** ([14], Theorem 17). *Let  $f \in \mathcal{S}(\Omega_D)$  and let  $x = \alpha + J\beta \in \Omega_D$ . Then one of the following mutually exclusive statements holds:*

- (1)  $\mathbb{S}_x \cap V(f) = \emptyset$ .
- (2)  $\mathbb{S}_x \subset V(f)$ . In this case  $x$  is called a real or spherical zero of  $f$  if, respectively,  $x \in \mathbb{R}$  or  $x \notin \mathbb{R}$ .
- (3)  $\mathbb{S}_x \cap V(f) = \{y\}$ , with  $y \notin \mathbb{R}$ . In this case  $x$  is called an  $\mathbb{S}$ -isolated non-real zero of  $f$ .

**Proposition 10** ([14], Proposition 25). *Let  $f, g \in \mathcal{S}(\Omega_D)$ . Then  $V(f) \subset V(f \cdot g)$ . Moreover it holds:*

$$\bigcup_{x \in V(f \cdot g)} \mathbb{S}_x = \bigcup_{x \in V(f) \cup V(g)} \mathbb{S}_x.$$

**Corollary 11** ([14], Corollary 19). *If  $f$  is a real slice function then  $f$  does not have  $\mathbb{S}$ -isolated non-real zeros. Moreover, for any slice function  $f$ , it holds:*

$$V(N(f)) = \bigcup_{x \in V(f)} \mathbb{S}_x.$$

In the next theorem we add regularity property.

**Theorem 12** ([14], Theorem 20). *Let  $\Omega_D$  be a connected circular domain and let  $f$  be a slice regular function such that  $N(f)$  does not vanish identically, then  $V(f) \cap D_J$  is closed and discrete in  $D_J$  for every  $J \in \mathbb{S}$ .*

In particular in [2, 12, 19] it is stated an Identity Principle.

**Theorem 13** (Identity principle). *Let  $\Omega_D \subset \mathbb{H}$  be a connected domain and let  $f : \Omega_D \rightarrow \mathbb{H}$  be a slice regular function.*

- ([19], **Proposition 3.3**) *If  $\Omega_D \cap \mathbb{R} \neq \emptyset$  and if there exists  $I \in \mathbb{S}$  such that  $D_I \cap V(f)$  has an accumulation point, then  $f \equiv 0$  on  $\Omega_D$ .*
- ([2], **Theorem 3.6**) *If  $\Omega_D \cap \mathbb{R} = \emptyset$  and if there exist  $K \neq J \in \mathbb{S}$  such that both  $D_K^+ \cap V(f)$  and  $D_J^+ \cap V(f)$  contain accumulation points, then  $f \equiv 0$  on  $\Omega_D$ .*

The distinction between the two cases in the previous theorem is underlined by the next example.

**Example 1.** The slice regular function defined on  $\mathbb{H} \setminus \mathbb{R}$  by

$$f(x) = 1 - Ii, \quad x = \alpha + \beta I \in \mathbb{C}_I^+$$

is induced by a locally constant stem function and its zero set  $V(f)$  is the half plane  $\mathbb{C}_{-i}^+ \setminus \mathbb{R}$ . The function can be obtained by the representation formula in Theorem 2 by choosing the constant values 2 on  $\mathbb{C}_i^+ \setminus \mathbb{R}$  and 0 on  $\mathbb{C}_{-i}^+ \setminus \mathbb{R}$ .

The notion of slice constant function was introduced in [2] to isolate the class of functions for which the previous example is a representative.

**Definition 9** ([2]). Let  $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ .  $f$  is called *slice constant* if the stem function  $F$  is locally constant.

**Proposition 14** ([2], Theorem 3.4). *Let  $f \in \mathcal{S}(\Omega_D)$  be a slice constant function, then  $f$  is slice regular. Moreover  $f$  is slice constant if and only if*

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right) \equiv 0.$$

The next definition is needed for defining the multiplicity of a slice function at a point.

**Definition 10** ([14]). The characteristic polynomial of  $y$  is the slice regular function  $\Delta_y(x) : \mathbb{H} \rightarrow \mathbb{H}$  defined by:

$$\Delta_y(x) := N(x - y) = (x - y) \cdot (x - y^c) = x^2 - x(y + y^c) + yy^c.$$

**Remark 7.** The following facts about the characteristic polynomial are quite obvious. We refer the reader to [14], Section 7.2.

- $\Delta_y$  is a real slice function.
- Two characteristic polynomials  $\Delta_y, \Delta_{y'}$  coincide if and only if  $\mathbb{S}_y = \mathbb{S}_{y'}$ .
- $V(\Delta_y) = \mathbb{S}_y$ .

It is showed in [14], Corollary 23, that, if  $f$  belongs to  $\mathcal{SR}(\Omega_D)$  and  $x_0 \in V(f)$ , then  $\Delta_{x_0}(x)$  divides  $N(f)$ . Thanks to this fact we are able to give the following definition.

**Definition 11** ([14]). Let  $f \in \mathcal{SR}(\Omega_D)$  such that  $N(f)$  does not vanish identically. Given  $n \in \mathbb{N}$  and  $x_0 \in V(f)$ , we say that  $x_0$  is a zero of  $f$  of *total multiplicity*  $n$ , and we will denote it by  $m_f(x_0)$ , if  $\Delta_{x_0}^n \mid N(f)$  and  $\Delta_{x_0}^{n+1} \nmid N(f)$ . If  $m_f(x_0) = 1$ , then  $x_0$  is called a *simple zero* of  $f$ .

The last definition, stated in [14], is equivalent to the one of *total multiplicity* stated in [13, 10]. The adjective “total” was introduced to underline the fact that this integer take into accounts both spherical and isolated order of zero of a point. We will use this adjective in this paper to distinguish the last notion of multiplicity to the one stated at the end of this section.

We recall now the definition of the degenerate set of a function.

**Definition 12** ([10]). Let  $f \in \mathcal{S}(\Omega_D)$  and let  $x = \alpha + I\beta \in \Omega_D$ ,  $\beta > 0$  be such that  $\mathbb{S}_x = \alpha + \mathbb{S}\beta \subset \Omega_D$ . The 2-sphere  $\mathbb{S}_x$  is said to be *degenerate* for  $f$  if the restriction  $f|_{\mathbb{S}_x}$  is constant. The union  $D_f$  of all degenerate spheres for  $f$  is called *degenerate set* of  $f$ .

Observe that the degenerate set of a slice function is a circular domain. We will now state some properties of the degenerate set of a slice function. First of all, the degenerate set of a slice function can be described as the zero set of the spherical derivative as stated in the following proposition.

**Proposition 15** ([2], Proposition 4.11). *Let  $f$  be a slice function over  $\Omega_D$ , then we have the following equality:*

$$D_f = V(\partial_s f).$$

Moreover  $D_f$  is closed in  $\Omega_D \setminus \mathbb{R}$ .

*Proof.* The proof of the statement is trivial thanks to Remark 3. □

As usual, adding the regularity property implies several additional results as the following one.

**Proposition 16** ([2], Proposition 4.12). *If  $f \in \mathcal{SR}(\Omega_D)$  is non-constant, then the interior of  $D_f$  is empty.*

In the next theorem and proposition we say something more about the zero set of a slice regular function. These results were exposed for the first time in the Ph.D. thesis [1] of the author and deal with the possibility, for a slice regular function  $f : \Omega_D \rightarrow \mathbb{H}$  with  $\Omega_D \cap \mathbb{R} = \emptyset$ , to admit surfaces  $S_f \subset \Omega_D$  on which is constant.

**Theorem 17.** *Let  $\Omega_D$  be a connected circular domain such that  $\Omega_D \cap \mathbb{R} = \emptyset$ . Let  $f \in \mathcal{SR}(\Omega_D)$  be a non-constant function. If  $x_0 \in V(f)$  is not isolated in  $V(f)$ , then there exists a real surface  $\mathcal{S} \subset \Omega_D$  such that  $x_0 \in \mathcal{S} \subset V(f)$ . Moreover,  $V(f)$  does not contain any 3-manifold  $M$ .*

*Proof.* Let us start with the three dimensional case: writing  $\Omega_D$  as the product  $D^+ \times \mathbb{S}$ , we have that, if  $V(f)$  contains a three-dimensional manifold  $M$ , then it can be split as  $M_D \times M_{\mathbb{S}}$ , with  $M_D \subset D^+$  and  $M_{\mathbb{S}} \subset \mathbb{S}$ . Since  $M$  has dimension 3, then, it is not fully contained either in  $D^+$  or in  $\mathbb{S}$  and it must contain either an open set of  $D^+$  times a curve in  $\mathbb{S}$  or, conversely, an open set of  $\mathbb{S}$  times a curve in  $D^+$ . But if  $M$  contains an open subset of  $\Lambda \subset D^+$  times a curve in  $\mathbb{S}$ , then there are at least two imaginary units  $I \neq J \in \mathbb{S}$  such that, denoting by  $\Lambda_K^+$  the projection of  $\Lambda$  in  $D_K^+$ ,  $f(\Lambda_I^+) = f(\Lambda_J^+) \equiv 0$  and so, thanks to Theorem 13,  $f \equiv 0$ . In the other case, if  $M$  contains an open set in  $\mathbb{S}$  times a curve in  $D^+$ , then, thanks to Theorem 2,  $M$  contains the whole sphere and so  $f$  is equal to zero on a curve contained in  $D^+$  of degenerate spheres. Fixing then two different imaginary units  $I \neq J \in \mathbb{S}$  we have that  $f_I^+$  and  $f_J^+$  are identically zero on a curve and so, again,  $f \equiv 0$ . So, eventually,  $V(f)$  cannot contain a 3D-manifold.

Let now  $f = \mathcal{I}(F_1 + \sqrt{-1}F_2)$ ,  $x = \alpha + I\beta \in \Omega_D$  and  $z = \alpha + i\beta \in D^+$ . If  $x$  is an accumulation point in  $V(f) \cap \mathbb{S}_x$  then it is clear that the whole sphere  $\mathbb{S}_x$  is contained in the zero locus of  $f$ . Analogously, if  $x$  is an accumulation point for  $V(f) \cap D_I^+$ , then  $D_I^+ \subset V(f)$ . Let us consider then the case in which  $x$  is a generic accumulation point that doesn't accumulate in any sphere or in any semislice. The point  $x$  belongs to  $V(f)$  if and only if  $F_1(z) + IF_2(z) = 0$ . Since  $x$  doesn't accumulate in any sphere that intersects  $V(f)$ , then  $F_2(z) \neq 0$ . Then the zero locus of  $f$  is equal to

$$V(f) = V(\partial_s f) \sqcup \{x \in D^+ \times \mathbb{S} \mid x = (z, -F_1(z)F_2(z)^{-1}) \in D^+ \times \mathbb{S}\}.$$

Since  $x$  is an accumulation point in  $V(f) \setminus V(\partial_s f)$ , this means that, for any open disc centered in  $z$  and contained in  $D^+$ , there are infinite points  $w$  such that there exists an imaginary unit  $I_w$  for which  $F_1(w) + I_w F_2(w) = 0$ . Hence, for any  $J \in \mathbb{S}$ , the normal function  $N(f)$  restricted to  $D_J^+$  vanishes at infinite points that accumulates to  $\alpha + J\beta$  and so, for the Identity Principle,  $N(f) \equiv 0$ . So, for any  $z \in D^+$ , there exists  $I_z \in \mathbb{S}$ , such that  $(z, I_z) \in (D^+ \times \mathbb{S}) \cap V(f)$ . Now, the condition  $N(f) \equiv 0$ , translates in the following system

$$(1) \quad \begin{cases} g(F_1, F_1) - g(F_2, F_2) = 0 \\ g(F_1, F_2) = 0. \end{cases}$$

System 1 implies that, for any  $z \in D^+$ ,  $\| -F_1(z)F_2(z)^{-1} \| = 1$  and the real part  $\operatorname{Re}(F_1(z)F_2(z)^{-1}) = 0$  and so  $F_1(z)F_2(z)^{-1} \in \mathbb{S}$ . Finally, the set

$$\widetilde{V(f)} = \{x \in D^+ \times \mathbb{S} \mid x = (z, -F_1(z)F_2(z)^{-1})\}$$

defines a real surface in  $D^+ \times \mathbb{S}$  that contains the accumulation point  $x$ . □

Therefore, the zero locus of a non-constant slice regular function  $f$ , contains isolated points, null-spheres and generic surfaces (possibly semislices), not contained in the degenerate set.

**Proposition 18.** *Given a connected circular open domain  $\Omega_D$  and a slice regular function  $f \in \mathcal{SR}(\Omega_D)$ , if there exist  $q \in \mathbb{H}$  such that  $h = f - q$  admits two different surfaces  $S_1$  and  $S_2$  that are not degenerate spheres, in the zero locus (i.e.:  $S_1, S_2 \subset V(f)$ ), then  $f$  is constant.*

*Proof.* Without loss of generality, we can suppose  $q = 0$ . Then, for any  $z \in D^+$  there exist  $I_1 \neq I_2 \in \mathbb{S}$  such that  $f$  vanishes both at  $(z, I_1)$  and  $(z, I_2)$  in  $\Omega_D = D^+ \times \mathbb{S}$ . This will imply that the spherical derivative is everywhere equal to zero and so  $f$  is constant. □

**Remark 8.** The condition  $N(f) \equiv 0$  defines a surface in  $\Omega_D$  that can coincide with a semislice  $D_I^+$ , for some  $I \in \mathbb{S}$ , or not. We will see in the next pages (see Lemma 30), that the set of surfaces in which a slice regular function is constant is contained in a possibly bigger set that is closed and with empty interior.



**Example 2.** Let  $g : \mathbb{H} \rightarrow \mathbb{H}$  be the slice regular function defined by  $g(x) = x + j$  and let  $f$  be the slice regular function defined in example 1. Consider now the slice regular function  $h : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$  defined by  $h := g \cdot f$ . Explicitly, this function is defined by

$$h(\alpha + I\beta) = \alpha + \beta i + j + I(\beta - \alpha i + k),$$

where  $\alpha + I\beta$  belongs to  $\mathbb{H} \setminus \mathbb{R}$ . The spherical derivative of  $h$  is equal to

$$\partial_s h(\alpha + I\beta) = 1 - \frac{\alpha}{\beta} i + \frac{k}{\beta},$$

that is always nonzero. Then, the function  $h$  is not constant in any sphere. We want to look for the zero set of  $h$  and then we have to impose the following equation:

$$h(\alpha + I\beta) = \alpha + \beta i + j + I(\beta - \alpha i + k) = 0$$

We remember Proposition 10 which says that the zero set of the product  $h = g \cdot f$  is composed by the union of the zero set of  $g$  with the zero set of  $f$  “properly modified” (this “modification” is given by the formula in Proposition 7). We have then that  $h(-j) = 0$ . Suppose, now,  $x \neq -j$ . Then  $h(x) = 0$  if and only if

$$I = \frac{-(\alpha^2 + \beta^2 - 1)i - 2\beta j + 2\alpha k}{\alpha^2 + \beta^2 + 1}.$$

But then, the surface  $S_h : \mathbb{C}^+ \rightarrow \mathbb{H} \setminus \mathbb{R}$  defined by,

$$S_h(\alpha + i\beta) = \left( \alpha + i\beta, \frac{-(\alpha^2 + \beta^2 - 1)i - 2\beta j + 2\alpha k}{\alpha^2 + \beta^2 + 1} \right) \subset \mathbb{C}^+ \times \mathbb{S} \simeq \mathbb{H} \setminus \mathbb{R},$$

is a “non trivial” surface (i.e.: not a sphere nor a semislice), on which the slice regular function  $h$  result to be constant and equal to zero. Observe that  $-j$  is in the image of  $S_h$ , in fact, for  $S_h(i) = (i, -j)$ , therefore  $V(h) = S_h$ .

However, this surface is not the only 2-dimensional manifold contained in the domain of  $h$  on which the function is constant. The function  $h$  is, indeed, constant and equal to  $2j$  on the semislice  $\mathbb{C}_{-i}^+$ . This was suggest by the fact that the slice derivative of  $h$  is equal to  $\partial h / \partial x(\alpha + I\beta) = 1 - Ii = f(\alpha + I\beta)$ .

Later we will see that these are the only surfaces on which this function is constant.

**Remark 9.** If  $\Omega_D \cap \mathbb{R} \neq \emptyset$ , then the fact that a slice regular function  $f : \Omega_D \rightarrow \mathbb{H}$  is such that  $N(f) \equiv 0$ , implies that  $f \equiv 0$  (see, e.g., [14], Theorem 20). Then, if there exists a sphere  $\mathbb{S}_x \subset \Omega_D$  such that the zeros of  $f$  in  $\Omega_D \setminus \mathbb{S}_x$  accumulates to a point of  $\mathbb{S}_x$ , this implies that  $f \equiv 0$  (cf [10], Corollary 3.14).

The last introductory tool come from complex analysis. The main reference for the following is [17], Chapter V.9.

**Definition 13.** Given a holomorphic function  $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  we define the *multiplicity* of  $f$  at a point  $x \in D$  as the number:

$$n(x; f) := \inf \{k \in \mathbb{N} \setminus \{0\} \mid f^{(k)}(x) \neq 0\},$$

$f^{(k)}(x)$  denoting the  $k^{th}$  derivative of  $f$  with respect to  $z$  evaluated in  $x$ .

**Definition 14.** Given a holomorphic function  $f$  defined over a region  $D$  we define the *valence* of  $f$  at  $w \in \mathbb{C} \cup \{\infty\}$  as

$$v_f(w) := \begin{cases} +\infty & \text{if the set } \{f(z) = w\} \text{ is infinite;} \\ \sum_{f(z)=w} n(z; f) & \text{otherwise.} \end{cases}$$

If  $f$  does not take the value  $w$ , then  $v_f(w)$  is obviously equal to zero.

**Remark 10.** If  $f$  is a holomorphic function on a region  $D$  and is not constant, then for any  $r > 0$ , such that  $\overline{D(x; r)} \subset D$ , the valence at  $w$  of  $f|_{D(x; r)}$  is constant on each component of  $(\mathbb{C} \cup \{\infty\}) \setminus f(\partial D(x; r))$ , where  $D(x; r)$  denote the disc centered in  $x$  of radius  $r$ .

### 3. SLICE DIFFERENTIAL OF A SLICE FUNCTION

This section contains material and ideas that were introduced by the author in [1], we hope to further develop in the future.

Let  $x \in \mathbb{H} \simeq \mathbb{R}^4$ ,  $x = (x_0, x_1, x_2, x_3)$  with  $(x_1, x_2, x_3) \neq (0, 0, 0)$  (i.e.:  $x \in \mathbb{H} \setminus \mathbb{R}$ ). When we talk about slice functions we implicitly use the following change of coordinates:

$$(x_0, x_1, x_2, x_3) \mapsto (\alpha, \beta, I),$$

where  $\alpha \in \mathbb{R}$ ,  $\beta > 0$  and  $I = I(\vartheta, \varphi) \in \mathbb{S}$  with the following equalities:

$$\begin{cases} \alpha &= x_0 \\ \beta &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \vartheta &= \arccos(\frac{x_3}{\beta}) \\ \varphi &= \arctan(\frac{x_2}{x_1}). \end{cases}$$

Let now  $f : \Omega \subset \mathbb{R}^4 \rightarrow \mathbb{R}^4$  be any differentiable function. Then, its differential in these new coordinates, can be written in its domain, as follows

$$(2) \quad df = \left( \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta \right) + \frac{1}{\beta} \left( \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \varphi} d\varphi \right),$$

where :

$$\begin{cases} d\alpha &= dx_0 \\ d\beta &= \sin \vartheta \cos \varphi dx_1 + \sin \vartheta \sin \varphi dx_2 + \cos \vartheta dx_3 \\ d\vartheta &= \cos \vartheta \cos \varphi dx_1 + \cos \vartheta \sin \varphi dx_2 - \sin \vartheta dx_3 \\ d\varphi &= -\sin \varphi dx_1 + \cos \varphi dx_2. \end{cases}$$

We would like, however, to consider also  $\beta < 0$  (having in mind that a non-real quaternion  $x$  can be written both as  $\alpha + I\beta$  and  $\alpha + (-I)(-\beta)$ ). But in this case we have to take care that  $d\beta(-\beta, I) = d\beta(\beta, -I) = -d\beta(\beta, I)$ .

The aim of this section is to study the first part of the right hand side of equation 2, when  $f$  is a  $\mathcal{C}^1$  slice function.

We will start with the following general definition.

**Definition 15.** Let  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$ . We define the *slice differential*  $d_{sl}f$  of  $f$  as the following differential form:

$$\begin{aligned} d_{sl}f : (\Omega_D \setminus \mathbb{R}) &\rightarrow \mathbb{H}^*, \\ \alpha + I\beta &\mapsto dF_1(\alpha + i\beta) + IdF_2(\alpha + i\beta). \end{aligned}$$

**Remark 11.** The one-form  $\omega : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}^*$  defined as  $\omega(\alpha + I\beta) = Id\beta$ , represents the outer radial direction to the sphere  $\mathbb{S}_x = \{\alpha + K\beta \mid K \in \mathbb{S}\}$ . Then  $\omega(\alpha + I(-\beta)) = \omega(\alpha + (-I)\beta) = -\omega(\alpha + I\beta)$ . We can translate this observation in the language of slice forms. The function  $h(x) = Im(x)$  is a slice function induced by  $H(z) = \sqrt{-1}Im(z)$ . Then we have  $d_{sl}h(\alpha + I\beta) = Id\beta(\alpha + i\beta)$  and, thanks to the previous considerations  $d_{sl}h(\alpha + (-I)(-\beta)) = -Id\beta(\alpha - i\beta) = Id\beta(\alpha + i\beta)$ . Summarizing, we have that  $d\beta(\bar{z}) = -d\beta(z)$ . The same doesn't hold for  $d\alpha$  which is a constant one form over  $\mathbb{H}$  and for this reason in the next computations we will omit the variable (i.e.:  $d\alpha = d\alpha(z) = d\alpha(\bar{z})$ ).

We can show now that the previous definition is well posed.

**Proposition 19.** *Definition 15 is well posed, i.e. if  $D$  is symmetric with respect to the real axis, then*

$$d_{sl}f(\alpha + I\beta) = d_{sl}f(\alpha + (-I)(-\beta)), \quad \forall \alpha + I\beta \in \Omega_D \setminus \mathbb{R}$$

*Proof.* Let  $x = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$  and  $z = \alpha + i\beta$ , then,

$$\begin{aligned}
d_{sl}f(\alpha + (-I)(-\beta)) &= \\
&= \left( \frac{\partial F_1(\bar{z}) - IF_2(\bar{z})}{\partial \alpha} \right) d\alpha + \left( \frac{\partial F_1(\bar{z}) - IF_2(\bar{z})}{\partial \beta} \right) d\beta(\bar{z}) = \\
&= \frac{\partial F_1}{\partial \alpha}(\bar{z})d\alpha + \frac{\partial F_1}{\partial \beta}(\bar{z})(-1)d\beta(z) - I \left( \frac{\partial F_2}{\partial \alpha}(\bar{z})d\alpha + \frac{\partial F_2}{\partial \beta}(\bar{z})(-1)d\beta(z) \right) = \\
&= \frac{\partial F_1}{\partial \alpha}(z)d\alpha + \frac{\partial F_1}{\partial \beta}(z)d\beta(z) - I \left( -\frac{\partial F_2}{\partial \alpha}(z)d\alpha - \frac{\partial F_2}{\partial \beta}(z)d\beta(z) \right) = \\
&= \left( \frac{\partial F_1(z) + IF_2(z)}{\partial \alpha} \right) d\alpha + \left( \frac{\partial F_1(z) + IF_2(z)}{\partial \beta} \right) d\beta(z) = \\
&= d_{sl}f(\alpha + I\beta),
\end{aligned}$$

where the third equality holds thanks to the even-odd character of the couple  $(F_1, F_2)$ .  $\square$

To avoid ambiguity, in the following of this section we will consider always  $\beta > 0$ , so, to be more clear, the point  $p = \alpha - J\beta$  will be intended as  $p = \alpha + (-J)\beta$  and we will omit the argument of the one-form  $d\beta$ . We can represent, then, the slice differential as follows.

**Proposition 20.** *Let  $f = \mathcal{I}(F) \in \mathcal{S}^1(\Omega_D)$  with  $D \subset \mathbb{C}^+$  (so that  $\beta > 0$ ). Then, on  $\Omega_D \setminus \mathbb{R}$ , the following equality holds true.*

$$d_{sl}f = \frac{\partial f}{\partial \alpha}d\alpha + \frac{\partial f}{\partial \beta}d\beta.$$

*Proof.* The thesis follows from the following computations. Let  $x = \alpha + I\beta \in \Omega_D$  and  $z = \alpha + i\beta$ , then

$$\begin{aligned}
d_{sl}f(x) &= \left( \frac{\partial F_1}{\partial \alpha}(z)d\alpha + \frac{\partial F_1}{\partial \beta}(z)d\beta \right) + I \left( \frac{\partial F_2}{\partial \alpha}(z)d\alpha + \frac{\partial F_2}{\partial \beta}(z)d\beta \right) \\
&= \left( \frac{\partial F_1}{\partial \alpha}(z)d\alpha + I \frac{\partial F_2}{\partial \beta}(z)d\beta \right) + \left( \frac{\partial F_1}{\partial \beta}(z)d\beta + I \frac{\partial F_2}{\partial \alpha}(z)d\beta \right) \\
&= \frac{\partial f}{\partial \alpha}(x)d\alpha + \frac{\partial f}{\partial \beta}(x)d\beta.
\end{aligned}$$

$\square$

It is clear from the definition that, if we choose the usual coordinate system, where  $x = \alpha + I\beta$  with  $\beta > 0$ , then  $d_{sl}x = d\alpha + Id\beta$  and  $d_{sl}x^c = d\alpha - Id\beta$ . We can now state the following theorem.

**Theorem 21.** *Let  $f \in \mathcal{S}^1(\Omega_D)$ . Then the following equality holds:*

$$d_{sl}x \frac{\partial f}{\partial x}(x) + d_{sl}x^c \frac{\partial f}{\partial x^c}(x) = d_{sl}f(x), \quad \forall x \in \Omega_D \setminus \mathbb{R}.$$

*Proof.* The thesis is obtained after the following explicit computations:

$$\begin{aligned}
d_{sl}x \frac{\partial f}{\partial x} + d_{sl}x^c \frac{\partial f}{\partial x^c} &= \frac{1}{2} \left[ (d\alpha + Id\beta) \left( \frac{\partial F_1}{\partial \alpha} + \frac{\partial F_2}{\partial \beta} - I \left( \frac{\partial F_1}{\partial \beta} - \frac{\partial F_2}{\partial \alpha} \right) \right) + \right. \\
&\quad \left. + (d\alpha - Id\beta) \left( \frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta} + I \left( \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha} \right) \right) \right] \\
&= \frac{1}{2} \left[ d\alpha \frac{\partial F_1}{\partial \alpha} + d\alpha \frac{\partial F_2}{\partial \beta} - Id\alpha \frac{\partial F_1}{\partial \beta} + Id\alpha \frac{\partial F_2}{\partial \alpha} \right. \\
&\quad \left. + Id\beta \frac{\partial F_1}{\partial \alpha} + Id\beta \frac{\partial F_2}{\partial \beta} + d\beta \frac{\partial F_1}{\partial \beta} - d\beta \frac{\partial F_2}{\partial \alpha} + \right. \\
&\quad \left. + d\alpha \frac{\partial F_1}{\partial \alpha} - d\alpha \frac{\partial F_2}{\partial \beta} + Id\alpha \frac{\partial F_1}{\partial \beta} + Id\alpha \frac{\partial F_2}{\partial \alpha} + \right. \\
&\quad \left. - Id\beta \frac{\partial F_1}{\partial \alpha} + Id\beta \frac{\partial F_2}{\partial \beta} + d\beta \frac{\partial F_1}{\partial \beta} + d\beta \frac{\partial F_2}{\partial \alpha} \right] \\
&= d\alpha \frac{\partial F_1}{\partial \alpha} + Id\beta \frac{\partial F_2}{\partial \beta} + d\beta \frac{\partial F_1}{\partial \beta} + Id\alpha \frac{\partial F_2}{\partial \alpha} \\
&= \frac{\partial F_1}{\partial \alpha}d\alpha + \frac{\partial F_1}{\partial \beta}d\beta + I \left( \frac{\partial F_2}{\partial \alpha}d\alpha + \frac{\partial F_2}{\partial \beta}d\beta \right) \\
&= d_{sl}f.
\end{aligned}$$

□

We have then the obvious corollary:

**Corollary 22.** *Let  $f \in \mathcal{SR}(\Omega_D)$ . Then the following equality holds:*

$$d_{sl}x \frac{\partial f}{\partial x}(x) = d_{sl}f(x), \quad \forall x \in \Omega_D \setminus \mathbb{R}.$$

#### 4. THE REAL DIFFERENTIAL OF A SLICE FUNCTION

In this section we will describe the real differential of a slice function. For this purpose, in addition to what we already discussed in the previous pages, we will remember some results and constructions due to Caterina Stoppato (see [20]). Moreover, we will also use the concept of *spherical differential* that will be introduced right now.

Given  $f \in \mathcal{S}^1(\Omega_D)$  we have seen that it is possible to define its slice differential, considering, roughly speaking, the restriction of its real differential, outside of the real line, to each semislice. It is clear that this object does not exhaust the description of the real differential. What we are going to define here is the missing part.

**Definition 16.** Let  $f \in \mathcal{S}^1(\Omega_D)$ . We define the *spherical differential* of  $f$  as the following differentiable form:

$$d_{sp}f : \Omega_D \setminus \mathbb{R} \rightarrow \mathbb{H}^*, \\ d_{sp}f(\alpha + J\beta) := d_{\mathbb{R}}f(\alpha + J\beta) - d_{sl}f(\alpha + J\beta),$$

where  $d_{\mathbb{R}}f(\alpha + J\beta)$  denote the real differential of  $f$ .

We will give now a more explicit description of the spherical differential of a slice function. Starting from equation 2, we have that,

$$d_{sp}f = df_{\mathbb{R}} - d_{sl}f = \frac{1}{\beta} \left( \frac{\partial f}{\partial \vartheta} d\vartheta + \frac{1}{\sin \vartheta} \frac{\partial f}{\partial \varphi} d\varphi \right),$$

but since, for every  $\alpha + J\beta \in \Omega_D \setminus \mathbb{R}$ ,  $f$  depends on  $J = J(\vartheta, \varphi)$  in an affine way, then,

$$d_{sp}f = \frac{1}{\beta} \left( \frac{\partial J}{\partial \theta} d\theta + \frac{1}{\sin \theta} \frac{\partial J}{\partial \varphi} d\varphi \right) F_2.$$

But if  $g : \mathbb{H} \rightarrow \mathbb{H}$  is the identity function,  $(g(\alpha + I\beta) = \alpha + I\beta)$ , then

$$d_{\mathbb{R}}g|_{\mathbb{H} \setminus \mathbb{R}} = d\alpha + Id\beta + (dI)\beta = d_{sl}x + \frac{1}{\beta} \left( \frac{\partial I}{\partial \theta} d\theta + \frac{1}{\sin \theta} \frac{\partial I}{\partial \varphi} d\varphi \right) \beta = d_{sl}x + d_{sp}x,$$

and so,

$$d_{sp}f = d_{sp}x \partial_s f$$

It seems then that, if  $f \in \mathcal{SR}(\Omega_D)$ , then its real differential satisfies the following equation:

$$df|_{\Omega_D \setminus \mathbb{R}} = d_{sl}x \frac{\partial f}{\partial x} + d_{sp}x \partial_s f,$$

where, the position of the elements of the cotangent space is on the *left*.

As the reader could object, the previous are only formal considerations but, in the next pages everything will be proved in the case of slice regular functions (in particular see Corollary 27). We remember firstly the notion of spherical analyticity and its consequences.

**4.1. Coefficients of the spherical expansion.** In [16, 20], the authors introduce, in slightly different contexts, a spherical series of the form

$$f(x) = \sum_{n \in \mathbb{N}} \mathcal{S}_{y,n}(x) s_n,$$

that gave some interesting results. More precisely, for each  $m \in \mathbb{N}$  we define, the slice regular polynomial functions

$$\mathcal{S}_{y,2m}(x) := \Delta_y(x)^m, \quad \mathcal{S}_{y,2m+1}(x) := \Delta_y(x)^m(x - y).$$

Note that, since  $\Delta_y$  is a real slice function, then  $\Delta_y^m = \Delta_y^m$ . Series of type  $\sum_{n \in \mathbb{N}} \mathcal{S}_{y,n}(x) s_n$  have convergence sets that are always open with respect to the Euclidean topology. In particular, they are open with respect to the following *Cassini* pseudometric. If  $x, y \in \mathbb{H}$ , we set

$$u(x, y) := \sqrt{\|\Delta_y(x)\|}.$$

The function  $u$  turn out to be a pseudometric on  $\mathbb{H}$ , whose induced topology is strictly coarser than the Euclidean one. A  $u$ -ball of radius  $r$  centered in  $y$  will be denoted by  $U(y, R) := \{x \in \mathbb{H} \mid u(x, y) < R\}$ . In [20, 16] it is showed that the sets of convergence of series  $\sum_{n \in \mathbb{N}} \mathcal{S}_{y,n}(x) s_n$  are  $u$ -ball centered at  $y$  and it is proved a corresponding Abel Theorem (see fig. 8.1 in [10]). Moreover in [16], formulas for computing the coefficients are given. In this context, the following is the definition of analyticity. For the following definition we refer to [20, 16].

**Definition 17.** Given a function  $f : \Omega \rightarrow \mathbb{H}$  defined on a non-empty open circular subset  $\Omega$  in  $\mathbb{H}$ , we say that  $f$  is *u-analytic* or *spherical analytic*, if, for all  $y \in \Omega$ , there exists a non-empty  $u$ -ball  $U$  centered at  $y$  and contained in  $\Omega$ , and a series  $\sum_{n \in \mathbb{N}} \mathcal{S}_{y,n}(x) s_n$  with coefficients in  $\mathbb{H}$ , which converges to  $f(x)$  for each  $x \in U \cap \Omega$ .

We have the following expected result.

**Theorem 23.** *Let  $\Omega_D$  be a connected circular set and  $f : \Omega_D \rightarrow \mathbb{H}$  be any function. The following assertions hold.*

- (1) ([20], **Corollary 4.3**) *If  $D \cap \mathbb{R} \neq \emptyset$ , then  $f$  is a slice regular function if and only if  $f$  is a spherical analytic function.*
- (2) ([16], **Theorem 5.8**) *If  $D \cap \mathbb{R} = \emptyset$ , then  $f$  is a slice regular function if and only if  $f$  is a spherical analytic slice function.*

Given a slice regular function  $f \in \mathcal{SR}(\Omega_D)$ , the methods described in [20, 16] to compute its spherical coefficients  $\{s_n\}$  at a fixed point, allow a correct explanation and interpretation only for the first two (see, e.g., [16], formula 30):

$$\begin{aligned} s_1 &= \frac{1}{2} \text{Im}(y)^{-1} (f(y) - f(y^c)) = \partial_s f(y) \\ s_2 &= \frac{1}{2} \text{Im}(y)^{-2} (2 \text{Im}(y) \frac{\partial f}{\partial x}(y) - f(y) + f(y^c)), \end{aligned}$$

and in particular

$$s_1 + 2 \text{Im}(y) s_2 = \frac{\partial f}{\partial x}(y).$$

The following proposition, which has an independent interest, allows us to understand better the nature of  $s_2$ .

**Proposition 24.** *Let  $f \in \mathcal{SR}(\Omega_D)$  be a slice regular function, then the following formula holds:*

$$(3) \quad \frac{\partial f}{\partial x}(x) = 2 \text{Im}(x) \left( \frac{\partial}{\partial x} \partial_s f \right)(x) + \partial_s f(x), \quad \forall x = \alpha + J\beta \in \Omega_D.$$

*Proof.* Let  $F = F_1 + \sqrt{-1}F_2$  the inducing stem function of  $f$  and let  $x = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$  and  $z = \alpha + i\beta$ , then,

$$\frac{\partial f}{\partial x}(x) = \frac{1}{2} \left( \frac{\partial F_1}{\partial \alpha}(z) + J \frac{\partial F_2}{\partial \alpha}(z) - J \frac{\partial F_1}{\partial \beta}(z) + \frac{\partial F_2}{\partial \beta}(z) \right) = \circledast.$$

Using the slice regularity we have,

$$\circledast = \frac{\partial F_2}{\partial \beta}(z) + J \frac{\partial F_2}{\partial \alpha}(z) = 2J \left[ \frac{1}{2} \left( \frac{\partial F_2}{\partial \alpha}(z) - J \frac{\partial F_2}{\partial \beta}(z) \right) \right](x).$$

Now  $F_2(z) = \beta G(z)$ , with  $G = (F_2(z)/\beta)$  the stem function that induces the spherical derivative, then the last equation is equal to

$$\begin{aligned} & 2J \left[ \frac{1}{2} \left( \beta \frac{\partial G}{\partial \alpha}(z) - J\beta \frac{\partial G}{\partial \beta}(z) - JG(z) \right) \right] = \\ &= G(z) + 2J\beta \left( \frac{1}{2} \left( \frac{\partial G}{\partial \alpha}(z) - J \frac{\partial G}{\partial \beta}(z) \right) \right) \\ &= \partial_s f(x) + 2\operatorname{Im}(x) \left( \frac{\partial}{\partial x} \partial_s f \right)(x), \end{aligned}$$

where of course, in the last equality  $\partial_s f$  and  $\frac{\partial}{\partial x} \partial_s f$  are the slice functions induced by  $G$  and  $\frac{1}{2}(\frac{\partial G}{\partial \alpha} - J \frac{\partial G}{\partial \beta})$  respectively.

At this point we have proven the theorem in the case in which the point  $x$  is not real. Now, if the function  $f$  is defined also on the real line, then, thanks to slice regularity we have, in particular, that  $f$  is of class  $\mathcal{C}^\infty$ . Therefore, recalling Remark 3, we have that the spherical derivative and its slice derivative extends continuously to the real line and the proof of the theorem is concluded.  $\square$

**Remark 12.** Since the previous theorem holds for any  $x_0 \in \Omega_D$ , then, if  $x_0 \in \mathbb{R}$ , then we have that  $\frac{\partial f}{\partial x}(x_0) = \partial_s f(x_0)$ .

**Corollary 25.** Let  $f \in \mathcal{SR}(\Omega_D)$  be a slice regular function with spherical expansion  $f(x) = \sum_{n \in \mathbb{N}} \mathcal{S}_{y,n}(x) s_n$  centered in  $x_0 \in \Omega_D$  then,

$$s_2 = \frac{\partial}{\partial x}(\partial_s f)(x_0).$$

**4.2. Rank of the real differential of a slice regular function.** In [7, 20], the authors shown the following theorem.

**Theorem 26** ([20], Theorem 6.1). Let  $f \in \mathcal{SR}(\Omega_D)$  and  $x = \alpha + J\beta \in \Omega_D$ . For all  $v \in \mathbb{H}$ ,  $\|v\| = 1$ , it holds

$$\lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t} = v s_1 + (xv - vx^c) s_2,$$

where  $s_1$  and  $s_2$  are the first two coefficients of the spherical expansion of  $f$ .

The previous theorem has an important corollary (that is stated implicitly in [7], Section 3), which justifies explicitly the formal considerations in the introduction of Section 4.

**Corollary 27.** Let  $f \in \mathcal{SR}(\Omega_D)$  and let  $(df)_x$  denote the real differential of  $f$  at  $x = \alpha + J\beta \in \Omega_D$ . If we identify  $T_x \mathbb{H}$  with  $\mathbb{H} = \mathbb{C}_J \oplus \mathbb{C}_J^\perp$ , then for all  $v_1 \in \mathbb{C}_J$  and  $v_2 \in \mathbb{C}_J^\perp$ ,

$$(df)_x(v_1 + v_2) = v_1 \frac{\partial f}{\partial x}(x) + v_2 \partial_s f(x).$$

We will not give a proof of the previous theorem (and corollary) since the one in [20] does not use the additional hypothesis of nonempty intersection between the domain and the real axis. The only feature needed for the proof is, in fact, the existence, for every slice regular function, of a spherical expansion. But, as we state in Theorem 23, this is true also if the domain of definition of  $f$  does not intersects the real line (cf. [16], Theorem 1.8).

We now want to study the rank of a slice regular function. In [7] the authors proved that an injective slice regular function defined over a circular domain with real points, has invertible differential. The aim of the following pages is to extend this result to all slice regular functions. Let's start with a general result.

**Proposition 28** ([7], Proposition 3.3). Let  $f \in \mathcal{SR}(\Omega_D)$  and  $x_0 = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$ .

- If  $\partial_s f(x_0) = 0$  then:
  - $df_{x_0}$  has rank 2 if  $\frac{\partial f}{\partial x}(x_0) \neq 0$ ;
  - $df_{x_0}$  has rank 0 if  $\frac{\partial f}{\partial x}(x_0) = 0$ .
- If  $\partial_s f(x_0) \neq 0$  then  $df_{x_0}$  is not invertible at  $x_0$  if and only if

$$\frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} \in \mathbb{C}_J^\perp.$$

Let now  $\alpha \in \Omega_D \cap \mathbb{R}$ .  $df_{x_0}$  is invertible at  $\alpha$  if and only if its rank is not 0 at  $x_0 = \alpha + J\beta$ . This happens if and only if  $\partial_s f(x_0) \neq 0$ .

The proof of the previous statement can be found (with the appropriate change of notation), in [7] or in [10].

**Remark 13.** As the previous theorem states, the rank of a slice regular function is always an even number. This fact was pointed out in [18], Corollary 4, too.

**Definition 18.** Let  $f : \Omega \rightarrow \mathbb{H}$  any quaternionic function of quaternionic variable. We define the singular set of  $f$  as

$$N_f := \{x \in \Omega \mid df \text{ is not invertible at } x\}.$$

**Remark 14.** If a slice regular function  $f \in \mathcal{SR}(\Omega_D)$  is constant on a surface  $S$ , then  $S \subset N_f$ . This is obvious if  $S$  is in the degenerate set, but if  $S$  is not a degenerate sphere then this is true as well. If  $S$  is a semislice  $D_I^+$  for some  $I \in \mathbb{S}$ , then the slice derivative of  $f$  on that semislice is everywhere zero and so  $S \subset N_f$ . Suppose now that  $S$  is not in the degenerate set nor a semislice and  $f|_S \equiv 0$ . Then  $N(f) \equiv 0$  and this equality translates in the system in equation 1. Deriving the first equation of system 1 with respect to  $\beta$  and the second with respect to  $\alpha$  we obtain, for each  $z \in D$ ,

$$\begin{cases} g(\frac{\partial F_1}{\partial \beta}(z), F_1(z)) - g(\frac{\partial F_2}{\partial \beta}(z), F_2(z)) = 0 \\ g(\frac{\partial F_1}{\partial \alpha}(z), F_2(z)) + g(\frac{\partial F_2}{\partial \alpha}(z), F_1(z)) = 0. \end{cases}$$

If now  $x_0 = \alpha_0 + I_0\beta_0 \in S$  and  $z_0 = \alpha_0 + i\beta_0 \in D$ , then  $f(x_0) = 0$ , and so, if  $S$  is not degenerate,  $F_1(z_0) = -I_0F_2(z_0)$ . Evaluating the previous system in  $z_0$  we obtain

$$\begin{cases} g(\frac{\partial F_1}{\partial \beta}(z_0), -I_0F_2(z_0)) - g(\frac{\partial F_2}{\partial \beta}(z_0), F_2(z_0)) = 0 \\ g(\frac{\partial F_1}{\partial \alpha}(z_0), F_2(z_0)) + g(\frac{\partial F_2}{\partial \alpha}(z_0), -I_0F_2(z_0)) = 0, \end{cases}$$

and, using regularity and the fact that, for any  $p, q, r \in \mathbb{H}$ ,  $g(pq, r) = g(q, p^c r)$ , we get,

$$\begin{cases} g(I_0(\frac{\partial F_1}{\partial \beta}(z_0) + I_0\frac{\partial F_2}{\partial \beta}(z_0)), F_2(z_0)) = 0 \\ g(I_0(\frac{\partial F_1}{\partial \alpha}(z_0) + I_0\frac{\partial F_2}{\partial \alpha}(z_0)), F_2(z_0)) = 0, \end{cases}$$

and so

$$\begin{cases} \beta_0 \|\partial_s f(x_0)\| g(\frac{\partial f}{\partial x}(x_0)(\partial_s f)(x_0)^{-1}, 1) = 0 \\ \beta_0 \|\partial_s f(x_0)\| g(\frac{\partial f}{\partial x}(x_0)(\partial_s f)(x_0)^{-1}, I_0) = 0, \end{cases}$$

therefore, for any  $x_0 \in S$ , we have that  $x_0 \in N_f$ .

**Example 3.** We will compute now the singular set  $N_h$  of the function  $h : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$  defined in example 2,

$$h(\alpha + I\beta) = \alpha + \beta i + j + I(\beta - \alpha i + k), \quad \alpha + I\beta \in \mathbb{H} \setminus \mathbb{R}.$$

We have seen that  $\partial_s h \neq 0$  and so, thanks to Proposition 15,  $D_f = \emptyset$ . But then, as it is stated in Proposition 28, a point  $x \in \mathbb{H} \setminus \mathbb{R}$  belongs to  $N_h$  if and only if

$$\frac{\partial h}{\partial x}(x)(\partial_s h(x_0))^{-1} \in \mathbb{C}_I^\perp.$$

Based on the computations in example 2, the last can be written explicitly as,

$$(1 - Ii) \frac{(\beta + \alpha i - k)}{\beta \|\partial_s h(\alpha + I\beta)\|^2} \in \mathbb{C}_I^\perp.$$

After some computation, using the “scalar-vector” notation, we obtain that the previous relation is satisfied if and only if

$$\begin{cases} g(\frac{\partial f}{\partial x}(x)(\partial_s f)(x)^{-1}, 1) &= 0 \\ g(\frac{\partial f}{\partial x}(x)(\partial_s f)(x)^{-1}, I) &= 0 \end{cases} \Leftrightarrow \begin{cases} \beta(1 + g(I, i)) + g(I, j) &= 0 \\ \alpha(1 + g(I, i)) - g(I, k) &= 0 \end{cases}$$

This condition is clearly verified for any  $x \in \mathbb{C}_{-i}^+$  that is the semislice on which  $h$  is constant and equal to  $2j$ . Suppose then that  $I \neq -i$  and write  $I = Ai + Bj + Ck$ . The previous system become

$$\begin{cases} \beta &= -B(1+A)^{-1} \\ \alpha &= C(1+A)^{-1}, \end{cases}$$

and so, for any  $I \in \mathbb{S}$  such that  $-B(1+A)^{-1} > 0$  there exists a point  $\alpha + i\beta \in \mathbb{C}^+$  such that  $\alpha + I\beta \in N_h$ . We want to show now that the set of quaternions that satisfies these requirements is contained in the surface  $S_h$  defined in example 2. But if  $x = C(1+A)^{-1} - (Ai+Bj+Ck)B(1+A)^{-1}$ , then,

$$\begin{aligned} h(x) &= C(1+A)^{-1} - B(1+A)^{-1}i + \\ &\quad + j(Ai+Bj+Ck)(-B(1+A)^{-1} - C(1+A)^{-1}i + k) \\ &= C(1+A)^{-1} + AC(1+A)^{-1} - C + \\ &\quad + (-B(1+A)^{-1} - AB(1+A)^{-1} + B)i + \\ &\quad + (1-A-B^2(1+A)^{-1} - C^2(1+A)^{-1})j + \\ &\quad + (BC(1+A)^{-1} - BC(1+A)^{-1})k \\ &= 0, \end{aligned}$$

and since the zero set of  $h$  is exactly the surface  $S_h$  in example 2, then  $N_h = \mathbb{C}_{-i}^+ \cup S_h$ .

Since, now, the set of surfaces on which  $h$  is constant is contained in  $N_h$ , we obtain that  $\mathbb{C}_{-i}^+$  and  $S_h$  are the only two surfaces, contained in  $\mathbb{H} \setminus \mathbb{R}$  on which  $h$  is constant.

The following theorem will characterize the set  $N_f$  of singular points of  $f$ . In particular, the next theorem generalizes a well known concept in real and complex analysis i.e.: the fact that if the differential of a function is singular in some point  $x_0$ , then, the function can be expanded in a neighborhood of  $x_0$  as

$$f(x) = f(x_0) + o((x - x_0)^2).$$

**Theorem 29** ([7], Proposition 3.6). *Let  $f \in \mathcal{SR}(\Omega_D)$  and let  $x_0 = \alpha + \beta I \in \Omega_D$ . Then  $x_0 \in N_f$  if and only if there exists a point  $\widetilde{x}_0 \in \mathbb{S}_{x_0}$  and a function  $g \in \mathcal{SR}(\Omega_D)$  such that the following equation hold:*

$$f(x) = f(x_0) + (x - x_0) \cdot (x - \widetilde{x}_0) \cdot g(x).$$

*Equivalently,  $x_0 \in N_f$  if and only if the function  $f - f(x_0)$  has total multiplicity  $n \geq 2$  in  $\mathbb{S}_{x_0}$ .*

The proof of the last theorem in [7] does not use the hypothesis  $\Omega_D \cap \mathbb{R} \neq \emptyset$ . However, we will rewrite it in our setting with our notations. Before proving the last theorem we recall from [7] the following remark.

**Remark 15.** For all  $x_0 = \alpha + J\beta \in \mathbb{H} \setminus \mathbb{R}$ , setting  $\Psi(x) := (x - x_0)(x - x_0^c)^{-1}$  defines a stereographic projection of  $\alpha + \mathbb{S}\beta$  onto the plane  $\mathbb{C}_J^\perp$  from the point  $x_0^c$ . Indeed, if we choose  $K \in \mathbb{S}$  with  $K \perp J$  then for all  $x = \alpha + \beta L$  with  $L = tJ + uK + vJK \in \mathbb{S}$  we have  $\Psi(x) = (L - J)(L + J)^{-1} = \frac{u+vJ}{1+t}JK$  and  $\mathbb{C}_J \cdot K = (\mathbb{R} + \mathbb{R}J)JK = \mathbb{C}_J^\perp$ .

We are now able to pass to the proof of the theorem.

*Proof.* If  $x_0 \in \Omega_D \setminus \mathbb{R}$  then it belongs to  $D_f$  if and only if,  $f$  is constant on the sphere  $\mathbb{S}_{x_0}$ , i.e. there exists a slice regular function  $g : \Omega_D \rightarrow \mathbb{H}$  such that

$$f(x) - f(x_0) = \Delta_{x_0}(x)g(x).$$

This happens if and only if the coefficient  $s_1 = \partial_s f(x_0)$  in the spherical expansion vanishes.

Let now pass to the case  $x_0 \in \Omega_D \setminus \mathbb{R}$ ,  $x_0 \notin D_f$ . Thanks to Proposition 28,  $x_0 \in N_f$  if and only if,  $1 + 2Im(x_0)s_2s_1^{-1} = p \in \mathbb{C}_J^\perp$ . Thanks to the previous remark,  $p \in \mathbb{C}_J^\perp$  if and only if there exists  $\widetilde{x}_0 \in \mathbb{S}_{x_0} \setminus \{x_0^c\}$  such that  $p = (\widetilde{x}_0 - x_0)(\widetilde{x}_0 - x_0^c)^{-1}$ . The last formula is equivalent to

$$\begin{aligned} 2Im(x_0)s_2s_1^{-1} &= (\widetilde{x}_0 - x_0)(\widetilde{x}_0 - x_0^c)^{-1} - (\widetilde{x}_0 - x_0^c)(\widetilde{x}_0 - x_0)^{-1} \\ &= (\widetilde{x}_0 - x_0 - \widetilde{x}_0 + x_0^c)(\widetilde{x}_0 - x_0^c)^{-1} \\ &= -2Im(x_0)(\widetilde{x}_0 - x_0^c)^{-1}, \end{aligned}$$



that is  $s_1 = (x_0^c - \widetilde{x_0})s_2$ . Writing then the first terms of the spherical expansion of  $f$  around  $x_0$  we have:

$$\begin{aligned}
f(x) &= s_0 + (x - x_0)s_1 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x)(x - x_0) \cdot h(x) \\
&= s_0 + (x - x_0)(x_0^c - \widetilde{x_0})s_2 + \Delta_{x_0}(x)s_2 + \Delta_{x_0}(x) \cdot (x - x_0)h(x) \\
&= s_0 + (x - x_0)(x_0^c - \widetilde{x_0})s_2 + \\
&\quad + \Delta_{x_0}(x)s_2 + (x - x_0) \cdot (x - x_0^c) \cdot (x - x_0) \cdot h(x) \\
&= s_0 + (x - x_0) \cdot [(x_0^c - \widetilde{x_0} + x - x_0^c)s_2 + \Delta_{\widetilde{x_0}}(x)h(x)] \\
&= s_0 + (x - x_0) \cdot (x - \widetilde{x_0}) \cdot [s_2 + (x - \widetilde{x_0}^c)h(x)] \\
&= f(x_0) + (x - x_0) \cdot (x - \widetilde{x_0}) \cdot [s_2 + (x - \widetilde{x_0}^c)h(x)],
\end{aligned}$$

for some slice regular function  $h : \Omega_D \rightarrow \mathbb{H}$ , where we used the following facts:

- $(x - x_0)(x_0^c - \widetilde{x_0}) = (x - x_0) \cdot (x_0^c - \widetilde{x_0})$  because the second factor is constant;
- $\Delta_{x_0}(x)(x - x_0) = \Delta_{x_0}(x) \cdot (x - x_0)$  because the first factor is a real slice function;
- $(x - x_0^c) \cdot (x - x_0) = \Delta_{x_0}(x)$ ;
- $\Delta_{x_0}(x) = \Delta_{\widetilde{x_0}}(x)$  because  $\widetilde{x_0} \in \mathbb{S}_{x_0}$ .

Finally, if  $x_0 \in \Omega_D \cap \mathbb{R}$  then  $s_1 = 0$  if and only if

$$f(x) = f(x_0) + (x - x_0)^2 \cdot l(x) = f(x_0) + (x - x_0) \cdot (x - x_0) \cdot l(x),$$

for some slice regular function  $l : \Omega_D \rightarrow \mathbb{H}$ . □

For the main result we need, now, two lemmas.

**Lemma 30.** *Let  $f : \Omega_D \rightarrow \mathbb{H} \in \mathcal{SR}(\Omega_D)$  be non slice-constant. Then its singular set  $N_f$  is closed and with empty interior.*

*Proof.* Since  $D_f = V(\partial_s f)$  then it is closed in  $\Omega_D$ . So, since  $D_f \subset N_f$ , then the thesis is that  $N_f \setminus D_f$  is closed and has empty interior.

To show that  $N_f \setminus D_f$  is closed it is sufficient to observe that, for any  $y = \alpha + I\beta$  in this set

$$\frac{\partial f}{\partial x}(y) \partial_s f(y)^{-1} \in \mathbb{C}_I^\perp$$

and this is true if and only if

$$\frac{\partial f}{\partial x}(y) \partial_s f(y)^{-1} \text{Im}(y) = -\text{Im}(y) \frac{\partial f}{\partial x}(y) \partial_s f(y)^{-1}.$$

But then

$$N_f \setminus D_f = \{y \in \Omega_D \mid \frac{\partial f}{\partial x}(y) \partial_s f(y)^{-1} \text{Im}(y) + \text{Im}(y) \frac{\partial f}{\partial x}(y) \partial_s f(y)^{-1} = 0\},$$

and so it is the counter image, via a continuous function, of a closed set.

Let  $x_0 \in N_f \setminus D_f$  and *ad absurdum* let  $R > 0$  be a real number such that the open Euclidean ball  $B = B(x_0, R)$  centered in  $x_0$  with radius  $R$  is fully contained in  $N_f \setminus D_f$ . For any  $y \in B$  the spherical derivative  $\partial_s f(x_0) \neq 0$  and, by the previous Theorem 29, there exists a slice regular function  $h_y : \Omega_D \rightarrow \mathbb{H}$  such that  $N(f - f(y)) = \Delta_y(x)^2 h(x)$ , where  $N(f - f(y))$  is the normal function of  $f - f(y)$ . Computing the slice derivative of  $N(f - f(y))$  and evaluating in  $x = y$  we obtain

$$0 = \left[ \frac{\partial N(f - f(y))}{\partial x} \right]_{x=y} = \left[ \frac{\partial f}{\partial x} \cdot (f - f(y))^c \right]_{x=y}.$$

There are two cases 1)  $\frac{\partial f}{\partial x}(y) = 0$  or 2)  $\frac{\partial f}{\partial x}(y) \neq 0$ . Case 2) implies, using formula 7, that

$$f \left( \frac{\partial f}{\partial x}(y)^{-1} y \frac{\partial f}{\partial x}(y) \right) = f(y).$$

Case 1) can be divided into two sub-cases: *i*)  $y = \alpha + I\beta$  is an isolated zero for the slice derivative in  $D_f^+$  or *ii*)  $\frac{\partial f}{\partial x_I} \equiv 0$ . If *ii*) holds true, then we change our point  $y$  considering another point  $\omega \in B$  lying on another different semislice. Then,  $\omega$  can only be an isolated zero on its semislice for the slice derivative of  $f$  (otherwise  $f$  would be slice constant). The only possibility is, therefore, case *i*). If we are in case 1), *i*) then we can find a positive real number  $r$  such that the two dimensional

disc  $\Delta = \Delta_I(x_0, r)$  is contained in  $B \cap \mathbb{C}_I^+$  and, for any  $x \in \Delta \setminus \{y\}$  we have  $\frac{\partial f}{\partial x}(x) \neq 0$ . For any  $y' \in \Delta \setminus \{y\}$  we are in case 2) and, again, there are two sub cases: A)  $\frac{\partial f}{\partial x}(y')^{-1}y'\frac{\partial f}{\partial x}(y') \neq y'$  or B)  $\frac{\partial f}{\partial x}(y')^{-1}y'\frac{\partial f}{\partial x}(y') = y'$ . If there is a point that satisfies case A), then  $f$  would be equal to some quaternion  $p$  both in  $y'$  and in  $\frac{\partial f}{\partial x}(y')^{-1}y'\frac{\partial f}{\partial x}(y')$  and this would implies, using the representation theorem, that  $f|_{\mathbb{S}_y} \equiv p$  that is  $\mathbb{S}_y \in D_f$ . So, the only possible case is, finally, B). But if condition B) holds true for any  $y \in \Delta \setminus y'$ , then

$$y\frac{\partial f}{\partial x}(y) = \frac{\partial f}{\partial x}(y)y,$$

and so, for any  $y \in \Delta \setminus y'$ ,  $\frac{\partial f}{\partial x}(y)$  belongs to  $\mathbb{C}_I$  and so, thanks to Theorem 13, this is true for any point in  $D_I^+$ . We claim that this is not possible. In fact, if  $\alpha + I\beta = y \in B$ , then

$$\frac{\partial f}{\partial x}(y)\partial_s f(y)^{-1} \in \mathbb{C}_I^\perp$$

and, as before, this is true if and only if

$$(4) \quad \frac{\partial f}{\partial x}(y)\partial_s f(y)^{-1}Im(y) = -Im(y)\frac{\partial f}{\partial x}(y)\partial_s f(y)^{-1}.$$

But  $\frac{\partial f}{\partial x}(y)$  belongs to  $\mathbb{C}_I$  then it commutes with  $Im(y)$  and so, from the previous equation 4, we get,

$$\partial_s f(y)^{-1}Im(y) = -Im(y)\partial_s f(y)^{-1}$$

which means that  $\partial_s f(y) \in \mathbb{C}_I^\perp$  for each  $y \in D_I^+$ . This implies that there exists an imaginary unit  $J \in \mathbb{S}$  orthogonal to  $I$  and a function  $g : D_I^+ \rightarrow \mathbb{R}$  such that, for any  $y \in D_I^+$  it holds  $\partial_s f(y) = \frac{1}{\beta}g(y)J$ . Since the spherical derivative is independent from the imaginary unit  $I$  then it is  $g$  too. Since  $f = \mathcal{I}(F_1 + \sqrt{-1}F_2)$  is a slice regular function, then

$$\left(\frac{\partial f}{\partial x}\right)_I = \frac{\partial F_2}{\partial \beta} - I\frac{\partial F_2}{\partial \alpha} = \left(\frac{\partial g}{\partial \beta} - I\frac{\partial g}{\partial \alpha}\right)J$$

and this is not possible since, as we said, the slice derivative belongs to  $\mathbb{C}_I$ . □

**Lemma 31.** *Let  $f = \mathcal{I}(F) : \Omega_D \rightarrow \mathbb{H}$  be an injective slice function. Then for any  $x = \alpha + J\beta \in \Omega_D \setminus \mathbb{R}$ ,  $\partial_s f(x) \neq 0$ .*

*Proof.* We know that  $\partial_s f(x) = 0$  if and only if  $f$  is constant on the sphere  $\mathbb{S}_x$  (see Remark 3). But then if  $f$  is injective then  $\partial_s f(x) \neq 0$  for all  $x \in \Omega_D \setminus \mathbb{R}$ . □

Now we have that every injective slice regular function has real differential with rank at least equal to 2. The next step is to prove that for every injective slice regular function  $f$  the slice derivative  $\frac{\partial f}{\partial x}$  is everywhere different from 0.

**Theorem 32.** *Let  $f = \mathcal{I}(F) : \Omega_D \rightarrow \mathbb{H}$  be an injective slice regular function. Then its slice derivative  $\frac{\partial f}{\partial x}$  is always different from zero.*

*Proof.* What we want to prove is that, for any  $x_0 = \alpha + J\beta \in \Omega_D$

$$\frac{\partial f}{\partial x}(x_0) \neq 0.$$

First of all, thanks to the Identity Principle in Theorem 13 applied to the slice derivative of  $f$ , if  $\partial f/\partial x$  is equal to zero in  $y \in D_I \subset \Omega_D$ , for some  $I \in \mathbb{S}$ , then  $y$  is isolated in  $D_I$ . Since  $f$  is slice regular, then, thanks to Lemma 5, for any  $J \perp K \in \mathbb{S}$  there exist two holomorphic functions  $f_1, f_2 : D_J \rightarrow \mathbb{C}_J$  such that  $f_J = f_1 + f_2K$ .

Thanks to Lemma 3 we have then that,

$$\frac{\partial f}{\partial x}(x_0) = \frac{\partial f_1}{\partial z_J}(x_0) + \frac{\partial f_2}{\partial z_J}(x_0)K,$$

where  $\partial/\partial z_J = 1/2(\partial/\partial \alpha - J \cdot \partial/\partial \beta)$  and so, since  $f_1$  and  $f_2K$  lives on independent subspaces of  $\mathbb{H}$ , the thesis become that at least one of the two derivatives  $\frac{\partial f_1}{\partial z_J}(x_0)$ ,  $\frac{\partial f_2}{\partial z_J}(x_0)$  is different from zero.

Moreover, since  $f$  is injective, then also  $f_J$  is injective. So, if one between  $f_1$  and  $f_2$  is constant, then the other one must be injective, and so we will have an injective holomorphic function and the thesis will follow trivially. Let's suppose then that both  $f_1$  and  $f_2$  are non-constant functions and fix the following notations:

$$\begin{aligned} n(x; f) &:= \inf\{k \in \mathbb{N} \setminus \{0\} \mid \frac{\partial^k f}{\partial x^k}(x) \neq 0\}, \\ n_1(x; f) &:= \inf\{k \in \mathbb{N} \setminus \{0\} \mid f_1^{(k)}(x) \neq 0\}, \\ n_2(x; f) &:= \inf\{k \in \mathbb{N} \setminus \{0\} \mid f_2^{(k)}(x) \neq 0\}, \end{aligned}$$

where  $f_i^{(k)}$  denotes the  $k$ -th derivative of  $f_i$  with respect to  $\partial/\partial z_J$ . Using again Lemma 3, we have that, for every  $x \in D_J$ ,

$$n(x; f) = \min(n_1(x; f), n_2(x; f)).$$

Moreover, since  $f$  is non slice-constant then the null set of its slice derivative restricted to the semislice  $D_J$  is discrete. For this reason we can take two balls  $B_1 := B_1(x_0; r_1)$ ,  $B_2 := B_2(x_0; r_2)$ , such that their closure is contained in  $D_J^+$ ,  $f_i$  take the value  $f_i(x_0)$  on  $\overline{B_i}$  only at  $x_0$  and  $f_i'(z) \neq 0$  for any  $z \in B_i \setminus \{x_0\}$ . Let now  $B = B_1 \cap B_2$ , then, as it is pointed out in Remark 10, the valence  $v_{f_i}(f_i(z))$  of  $f_i|_B$  is constant and equal to  $n_i(z; f)$  in the component of  $(\mathbb{C}_J \cup \{\infty\}) \setminus f(\partial B)$  which contains  $f_i(z)$ . Since  $n(x; f) = \min(n_1(x; f), n_2(x; f))$  and  $n(x; f) = 1$  almost everywhere, then  $\exists y \in B$  and  $j \in \{1, 2\}$  such that  $1 = n(y; f) = n_j(y; f)$ . Then  $n_j$  is constant and equal to 1 in  $B$  and so  $f_j(\omega) \neq 0$ , for all  $\omega \in B$  and we have the thesis.  $\square$

**Remark 16.** The proof of the previous statement works also to prove that a slice regular function  $f : \Omega_D \rightarrow \mathbb{H}$  that is injective on a semislice  $D_J^+ \subset \Omega_D$  has slice derivative nonzero over the same semislice  $D_J^+$ . We choose to formalize the theorem in the previous less general hypothesis only to simplify the reading.

**Theorem 33.** *Let  $f$  be an injective slice regular function, then  $N_f = \emptyset$ .*

*Proof.* If, by contradiction, there exists  $x_0 = \alpha + J\beta \in N_f \neq \emptyset$ , then, thanks to Theorem 29, the function  $f - f(x_0)$  must have multiplicity  $n$  greater or equal to 2 at  $\mathbb{S}_{x_0}$ . This means that,

$$f(x) - f(x_0) = (x - x_0) \cdot g(x),$$

with  $g \in \mathcal{SR}(\Omega_D)$  such that  $g(x_1) = 0$  for some  $x_1 \in \mathbb{S}_{x_0}$ . Since  $f$  is injective, then  $g(x_0) = \frac{\partial f}{\partial x}(x_0) \neq 0$  and  $g(x_0^c) = \partial_s f(x_0) \neq 0$ , and so  $x_1 \neq x_0, x_0^c$ . Now, whereas we know the values of  $g$  at  $x_0$  and at  $x_0^c$ , we can apply the representation formula in Theorem 2 to analyze the behavior of  $f$  on the sphere  $\mathbb{S}_{x_0}$ . The result is the following,

$$g(\alpha + I\beta) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) - IJ \left( \frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0) \right) \right), \quad \forall I \in \mathbb{S}.$$

So, if there exist  $I \in \mathbb{S}$  such that  $g(\alpha + I\beta) = 0$ , then,

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0) + \partial_s f(x_0) &= IJ \left( \frac{\partial f}{\partial x}(x_0) - \partial_s f(x_0) \right) \\ \Leftrightarrow \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} + 1 &= IJ \left( \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} - 1 \right) \\ \Leftrightarrow \frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1} &= -(1 - IJ)^{-1}(1 + IJ), \end{aligned}$$

with  $I \neq J, -J$ , but then, since for  $I \neq \pm J$  the product  $-(1 - IJ)^{-1}(1 + IJ)$  has a non zero real part<sup>2</sup>, then  $\frac{\partial f}{\partial x}(x_0)(\partial_s f(x_0))^{-1}$  does not belong to  $\mathbb{C}_J^\perp$  and this is in contradiction with Proposition 28.  $\square$

<sup>2</sup>This can be viewed using the 'scalar-vector' notation.

**Example 4.** Let  $J \in \mathbb{S}$  be a fixed imaginary unit and  $f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H}$  be the slice regular function defined as,

$$f(\alpha + I\beta) = (\alpha + I\beta)(1 - IJ).$$

This function is constructed, by means of the representation formula, to be equal to zero over the semislice  $\mathbb{C}_{-J}^+$  and to be equal to  $2x$  over the opposite semislice  $\mathbb{C}_J^+$ . What we want to show is that the restriction  $f|_{\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}_{-J}^+)}$  is injective. This is trivial if we restrict the function to a semislice  $\mathbb{C}_I^+$ , so let  $x_1 = \alpha_1 + \beta_1 I_1 \neq \alpha_2 + \beta_2 I_2 = x_2$ , with  $I_1 \neq I_2$ , then

$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow \\ \Leftrightarrow x_1(1 - I_1 J) = x_2(1 - I_2 J) &\Leftrightarrow \\ \Leftrightarrow x_1 I_1 (I_1 + J) = x_2 I_2 (I_2 + J) &\Leftrightarrow \\ \Leftrightarrow (x_2 I_2)^{-1} (x_1 I_1) = -\frac{1}{c} (I_2 + J)(I_1 + J), \end{aligned}$$

where  $c = \|I_1 + J\|^2 \neq 0$ . Translating the variables  $x_1, x_2$  into their components, we obtain that, the last equality is equivalent to the following one:

$$-\frac{1}{\alpha_2^2 + \beta_2^2} [-\beta_1 \beta_2 + \alpha_1 \beta_2 I_1 - \alpha_2 \beta_1 I_2 + \alpha_1 \alpha_2 I_1 I_2] = -\frac{1}{c} [I_2 I_1 + I_2 J + J I_1 - 1].$$

Now we can decompose the last equation into the system involving the real and imaginary parts as follows:

$$\begin{cases} \frac{c}{\alpha_2^2 + \beta_2^2} [\beta_1 \beta_2 + \alpha_1 \alpha_2 I_2 \cdot I_1] = 1 + I_1 \cdot I_2 + (I_1 + I_2) \cdot J \\ \frac{c}{\alpha_2^2 + \beta_2^2} [\alpha_1 \beta_2 I_1 - \alpha_2 \beta_1 I_2 + \alpha_1 \alpha_2 I_2 \wedge I_1] = I_2 \wedge I_1 + (I_2 - I_1) \wedge J \end{cases}$$

where  $I \cdot J$  and  $I \wedge J$  denote the scalar and the vector products<sup>3</sup> respectively in  $\mathbb{R}^3$ . We will work now on the second equation of the previous system. Firstly, multiplying scalarly the equation by  $I_2 - I_1$ , we obtain that

$$\alpha_1 \beta_2 = -\alpha_2 \beta_1.$$

Substituting  $\alpha_1 = -\frac{\beta_1}{\beta_2} \alpha_2$  and multiplying scalarly by  $I_1 + I_2$  it follows that

$$(I_2 \wedge I_1) \cdot J = \frac{c}{2} \frac{\alpha_1 \beta_2}{\alpha_2^2 + \beta_2^2}.$$

Taking into account the previous results and multiplying scalarly by  $J$  and then by  $I_1$  (or  $I_2$ ), and supposing  $\alpha_2 \neq 0$ , we obtain the following two equalities:

$$(I_1 + I_2) \cdot J = -\frac{1}{2} \left[ 1 + c \frac{\alpha_2}{\beta_2} \frac{\alpha_2 \beta_1}{\alpha_2^2 + \beta_2^2} \right], \quad I_1 \cdot I_2 = -\frac{1}{2}.$$

Putting all these ingredients in the first equation of the system one obtain that:

$$\frac{c \beta_1}{\alpha_2^2 + \beta_2^2} \left[ \beta_2 + \frac{\alpha_2^2}{2 \beta_2} \right] = -\frac{1}{2} \frac{c \beta_1}{\alpha_2^2 + \beta_2^2} \frac{\alpha_2^2}{\beta_2},$$

and this is possible if and only if  $\beta_2^2 = -\alpha_2^2$ , which is absurd. If now  $\alpha_2 = 0$ , following the first part of the same argument, we obtain  $\alpha_1 = 0$  and so,

$$(5) \quad -\frac{c \beta_1}{\beta_2} = I_2 I_1 + I_2 J + J I_1 - 1.$$

But then, the imaginary part of  $I_2 I_1 + I_2 J + J I_1$ , that is  $I_2 \wedge I_1 + I_2 \wedge J + J \wedge I_1$ , must vanishes. This implies that  $(I_2 \wedge I_1) \cdot J = 0$  i.e.:  $J = A I_1 + B I_2$ , for some  $A$  and  $B$  real numbers both different from zero. In this case equation 5 becomes  $A + B + 1 - \frac{c \beta_1}{\beta_2} = (1 + A + B) I_1 I_2$  and so  $I_1 \wedge I_2 = 0$ . The last equalities (since  $I_1 \neq I_2$ ), entails  $I_1 = -I_2$  but this would imply  $\frac{\beta_1}{\beta_2} = 0$  and this is not possible.

<sup>3</sup>Here we used the 'scalar-vector' notation.

Since this function, with the proper restriction, is slice regular and injective then Theorem 33 says that its real differential is always invertible. This fact could also be seen computing the slice and the spherical derivative. Indeed, since

$$\partial_s f(\alpha + I\beta) = \frac{\beta - \alpha J}{\beta},$$

is always different from zero, we need only to control that the product  $\frac{\partial f}{\partial x}(\alpha + I\beta)(\partial_s f(\alpha + I\beta))^{-1}$  does not belong to  $\mathbb{C}_I^\perp$ . Now,

$$\frac{\partial f}{\partial x}(\alpha + I\beta)(\partial_s f(\alpha + I\beta))^{-1} = (1 - IJ) \left( \frac{\beta - \alpha J}{\beta} \right)^{-1} = \frac{\beta(1 - IJ)(\beta + \alpha J)}{\beta^2 + \alpha^2},$$

and so, whenever  $I \neq -J$ , the previous product has a nonzero real part and so does not belong to  $\mathbb{C}_I^\perp$ . The real differential of  $f$  can be represented in a point  $x = \alpha + I\beta$ , by means of slice and spherical forms, as,

$$df(\alpha + I\beta) = d_{sl}x(1 - IJ) + d_{sp}x \left( 1 - \frac{\alpha}{\beta}J \right).$$

**Remark 17.** In examples 2 and 3 we have studied the properties of the slice regular function  $h$  defined on  $\mathbb{H} \setminus \mathbb{R}$  as  $h(x) = (x + j) \cdot (1 - Ii) = x(1 - Ii) + (1 + Ii)j$ , where  $x = \alpha + I\beta$ . We found that this function admits two surfaces, namely  $S_h$  and  $\mathbb{C}_{-i}^+$ , on which it takes the constant values 0 and  $2j$ , respectively. These two surfaces, moreover, yields the singular set  $N_h$  of  $h$ . We want to show now, that, on  $(\mathbb{H} \setminus \mathbb{R}) \setminus (S_h \cup \mathbb{C}_{-i}^+)$  the function is injective. Take then  $I$  to be equal to  $Ai + Bj + Ck$  and  $x = \alpha + I\beta$ , then the function  $h$ , decomposed in its components with respect to  $1, i, j, k$ , is equal to,

$$h(x) = \alpha(A + 1) - C + (\beta(A + 1) + B)i + (\beta B - \alpha C + 1 - A)j + (\alpha B + \beta C)k.$$

First of all, since we are excluding the semislice  $\mathbb{C}_{-i}^+$ , we get  $(A + 1) \neq 0$ . Given  $q = q_0 + q_1i + q_2j + q_3k \notin \{0, 2j\}$ , we want to compute, where it is possible,  $h^{-1}(q)$ . So, we impose the system,

$$(6) \quad \begin{cases} \alpha(A + 1) - C = q_0 \\ \beta(A + 1) + B = q_1 \\ \beta B - \alpha C + 1 - A = q_2 \\ \alpha B + \beta C = q_3, \end{cases}$$

and, substituting  $\alpha = (q_0 + C)(1 + A)^{-1}$  and  $\beta = (q_1 - B)(1 + A)^{-1}$  in the last two equations and imposing  $A^2 + B^2 + C^2 = 1$ , we obtain that, for any  $q$  such that  $q_0^2 + q_1^2 \neq 0$ ,

$$(7) \quad \begin{cases} A = (q_0^2 + q_1^2 - q_2^2 - q_3^2)/\|q\|^2 \\ B = 2(q_0q_3 + q_1q_2)/\|q\|^2 \\ C = 2(q_1q_3 - q_0q_2)/\|q\|^2. \end{cases}$$

From the last system we get,

$$(8) \quad \begin{cases} \alpha = \frac{q_0\|q\|^2 + 2(q_1q_3 - q_0q_2)}{2(q_0^2 + q_1^2)} \\ \beta = \frac{q_1\|q\|^2 - 2(q_0q_3 + q_1q_2)}{2(q_0^2 + q_1^2)}. \end{cases}$$

If  $q_0^2 + q_1^2 = 0$ , then  $q_0 = 0 = q_1$  and then we get, from 6, that  $q_3 = 0$  as well. But then again, substituting  $C = \alpha(A + 1)$  and  $B = -\beta(A + 1)$  in the third equation and imposing  $A^2 + B^2 + C^2 = 1$ , we obtain that, the only possibility are  $q_2 = 0$  or  $q_2 = 2$ . At the end, what we get is that, the image of  $h$  is described as,

$$Im(h) = \left\{ q \in \mathbb{H} \mid q_0^2 + q_1^2 \neq 0, q_1 > \frac{2(q_0q_3 + q_1q_2)}{\|q\|^2} \right\} \cup \{0, 2j\},$$

moreover, since, for any  $q \in Im(h) \setminus \{0, 2j\}$  we can find only one counterimage  $h^{-1}(q) = \alpha + I\beta$ , given by the two systems in equations 7 and 8, then  $\tilde{h} = h|_{(\mathbb{H} \setminus \mathbb{R}) \setminus (S_h \cup \mathbb{C}_{-i}^+)}$  is injective and so

$N_{\tilde{h}} = \emptyset$ . Its real differential can be expressed in a point  $x = \alpha + I\beta$ , again by means of slice and spherical forms, as,

$$d\tilde{h}(\alpha + I\beta) = d_{sl}x(1 - Ii) + d_{sp}x \left(1 - \frac{\alpha}{\beta}i + \frac{k}{\beta}\right).$$

**Remark 18.** The reader could ask why we didn't follow the way of proving Theorem 33 by Gentili, Salamon and Stoppato in [7]. The answer is that, of course, that proof doesn't work in the case in which the domain of the function does not have real points. This fact, rather than being a mere observation, give space to interesting considerations that are not studied in this paper. To be precise, the theorem that fails is the following:

**Theorem 34.** *Let  $f : \Omega_D \rightarrow \mathbb{H}$  be a nonconstant regular function, and let  $\Omega_D \cap \mathbb{R} \neq \emptyset$ . For each  $x_0 = \alpha + I\beta \in N_f$ , there exists a  $n > 1$ , a neighborhood  $U$  of  $x_0$  and a neighborhood  $T$  of  $\mathbb{S}_{x_0}$  such that for all  $x_1 \in U$ , the sum of the total multiplicities of the zeros of  $f - f(x_1)$  in  $T$  equals  $n$ .*

A counter example, if the domain does not have real points, is given by the function,

$$f : \mathbb{H} \setminus \mathbb{R} \rightarrow \mathbb{H} \\ \alpha + I\beta \mapsto (\alpha + I\beta)(1 - IJ),$$

for a fixed  $J \in \mathbb{S}$ . As we have seen, this function is injective over  $\mathbb{H} \setminus (\mathbb{R} \cup \mathbb{C}_{-J}^+)$ , and so, if we take  $x_0 = -J \in N_f$ , for any neighborhood  $U$  of  $-J$  and any neighborhood  $T$  of  $\mathbb{S}_{-J}$  the sum of total multiplicities of the zeros of  $f - f(x_1)$ , for any  $x_1 \in U \setminus \mathbb{C}_{-J}^+$  is equal to 1. The previous function is constructed to be equal to 0 over  $\mathbb{C}_{-J}^+$  and equal to  $2x$  over  $\mathbb{C}_J^+$ , but other more complex examples can be build in this way, for example considering a function equal to some monomial  $x^m$  on a semislice and equal to another different monomial  $x^n$  on the opposite. This feature will certainly be a starting point for future investigations.

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